Dense Distance Magic Graphs

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Abstract: Let $G = (V, E)$ be a graph with n vertices. A bijection f from V to the set of integers $\{1, 2, \ldots, n\}$ is called a *distance magic labeling* of G if for every vertex in G the sum of labels of all adjacent vertices equas the same number k. A graph that allows such a labeling is a *distance magic graph*. For graphs with an even number of vertices there is an elegant construction of r-regular distance magic graphs for all feasible values of r. For graphs with an odd number of vertices some necessary and several sufficient conditions are known for a graph to have a distance magic labeling. In this paper we focus on distance magic graphs of high regularity: we provide constructions of $(n-5)$-regular distance magic graphs with n vertices. Magic labelings are used in tournament scheduling.

Keywords: labeling, distance magic, regular graph.

1 Introduction and definitions

All graphs in this paper are finite, undirected without loops and multiple edges. A *distance magic labeling* of a graph G with n vertices is a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ with the property that there exists an integer k such that for every vertex x is

$$w(x) = \sum_{y \in N(x)} f(y) = k,$$

where $N(x)$ is the set of all vertices adjacent to x. The constant k is the *magic constant* and $w(x)$ is the *weight* of vertex x. We say a graph is *distance magic* if it admits a distance magic labeling.

For convenience we identify vertices with their labels, e.g. vertex labeled 1 we call vertex 1. An example of a distance magic graph are in Figures 1 and 4.

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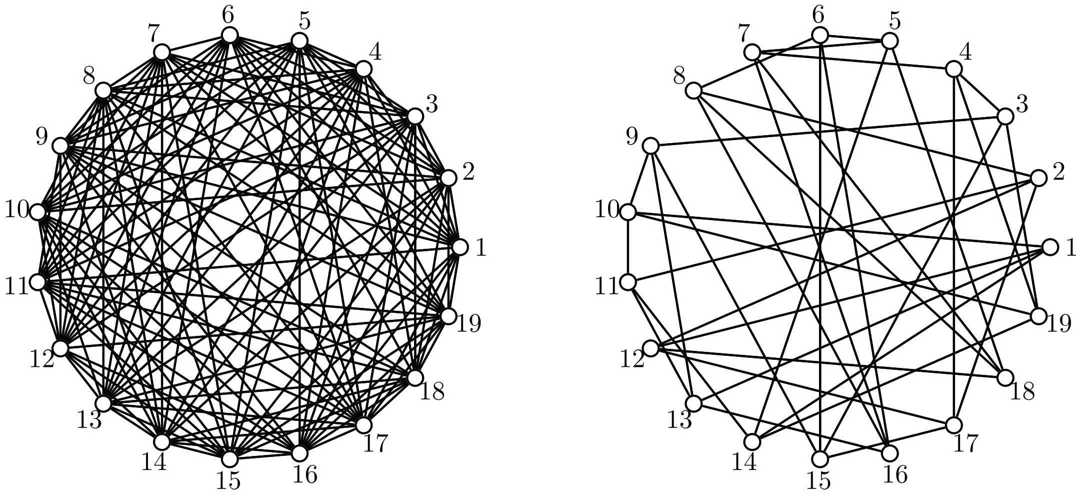

Figure 1: A 14-regular distance magic graph with 19 vertices and its complement.

2 Known results

The concept of distance magic labelings was introduced independently by several authors. Formerly, it was called 1-vertex magic vertex labelings [9] or Σ-labelling [5].

There is a survey on distance magic labelings by Arumugam, Fronček, and Kamatchi [1]. Here, we summarize results relevant to this paper, we focus on regular graphs. Most necessary conditions for a distance magic labeling of a given graph to exist are based on counting arguments and parity. Sufficient conditions involve constructions. For a distance magic labeling f of G with n vertices holds the following [9]:

$$nk = \sum_{x \in V(G)} \deg(x) f(x), \tag{1}$$

where $\deg(x)$ is the degree of vertex x. Thus, the magic constant of regular graphs is determined uniquely. In any r-regular distance magic graph G is

$$k = r(n+1)/2. \tag{2}$$

In [1], it was shown that the magic constant is determined uniquely for every distance magic graph.

From (2) it follows that a regular distance magic graphs can exist only if r is even or n is odd. The existence of distance magic labelings with an even number of vertices was completely settled by Fronček, Kovářová and the first author [3,4] who investigated the relation of regular distance magic graphs to scheduling of incomplete tournaments. The following proposition from [3] gives a necessary and sufficient condition for a distance magic graph with an even number of vertices to exist.

Proposition 2.1 *For n even a* $\mathrm{FIT}(n,k)$ *exists if and only if* $1 \le k \le n-1$, $k \equiv 1 \pmod 2$ *and either* $n \equiv 0 \pmod 4$ *or* $n \equiv k+1 \equiv 2 \pmod 4$.

An analogous simple condition for regular distance magic graphs with an odd number of vertices does not exist, because there are several exceptions for small orders. In [6] all orders of 4-regular distance magic graphs with an odd number of vertices were characterized; the smallest example has 17 vertices.

In [7] all $(n-3)$-regular distance magic graphs were characterized: they exists if and only if $n \equiv 3 \pmod 6$. Moreover, they are isomorphic to a balanced complete $n/3$-partite graph. In this paper we extend the results of Fronček [2] and of Silber and the first author [7].

In the next section we provide a construction of $(n-5)$-regular handicap graphs. For convenience, the construction describes the graph complement, which is a 4-regular graph with an *auxiliary* labeling. In an $(n-5)$-regular distance magic graph the weight of every vertex is $k = r(n+1)/2 = (n-5)(n+1)/2$. In a complete graph with vertices labeled 1 through n, the weight (obtained as the sum of adjacent vertices) is

$$w(i) = \sum_{i=1}^{n} i - i = n(n+1)/2 - i.$$

Thus, in the 4-regular complement of a $(n-5)$-regular distance magic graph the weight of every vertex is

$$w(i) = n(n+1)/2 - i - (n-5)(n+1)/2 = 5(n+1)/2 - i. \tag{3}$$

The constructive proof deals with several cases modulo 12, for each case a different distance magic graph is constructed. We provide details for one of the cases and only a basic idea for the remaining cases. In the conclusion we summarize the result in a single statement for all feasible orders.

The motivation for distance magic labeling comes from scheduling incomplete tournaments [1].

3 Kotzig array

Kotzig arrays are a generalization of magic rectangles [8]. In this paper we use the following array for $n = 2t+1$.

$$a_{i,j} = \begin{bmatrix} 1 & 2 & \cdots & t & t+1 & t+2 & t+3 & \cdots & 2t & 2t+1 \\ 4t+2 & 4t & \cdots & 2t+4 & 2t+2 & 4t+1 & 4t-1 & \cdots & 2t+5 & 2t+3 \\ 5t+3 & 5t+4 & \cdots & 6t+2 & 6t+3 & 4t+3 & 4t+4 & \cdots & 5t+1 & 5t+2 \end{bmatrix} \tag{4}$$

Notice, every integer $1, 2, \ldots, 3n = 6t+3$ appears in the array precisely once: $1, 2, \ldots, 2t+1$ in the first row, $2t+2, 2t+3, \ldots, 4t+2$ in the second row and $4t+3, 4t+4, \ldots, 6t+3$ in the third row. Moreover, the sum of every column is $9t+6$.

Lemma 3.1 *The sum of the first and the last entry of column of the Kotzig array (a_{ij}) containing $3t+2$ is $6t+4$. All remaining columns of the Kotzig array can be paired so that for the pair x, y of columns is $a_{1x} + a_{3y} = a_{2x} + a_{2y} = a_{3x} + a_{1y} = 6t+4$.*

Proof. The first part of the claim is obvious: since the sum of each column is $9t+6$, the sum of the remaining two elements in the column besides $3t+2$ is $6t+4$.

The proof of the second part follows. Wlog let $x < y$. For $x \in \{1, 2, \ldots, \lfloor t/2 \rfloor\}$ take $y = t+2-x$. Then $y \in \{2 + \lceil t/2 \rceil, \ldots, t+1\}$. Now $a_{1x} + a_{3y} = x + (6t+4-x) = 6t+4$, $a_{2x} + a_{2y} = (4t+4-2x) + (2t+2x) = 6t+4$, and $a_{3x} + a_{1y} = (5t+2+x) + (t+2-x) = 6t+4$. Similarly, for $x \in \{t+2, t+3, \ldots, \lfloor 3t/2 \rfloor\}$ take $y = 3t+3-x$. Again, one can check that $a_{1x} + a_{3y} = a_{2x} + a_{2y} = a_{3x} + a_{1y} = 6t+4$. $\square$

4 Orders of all $(n-5)$-regular distance magic graphs

In this section we show for which orders there exists an $(n-5)$-regular distance magic graph. The proof is constructive. As stated before, it suffices to give for the 4-regular complement $\overline{G}$ an auxiliary labeling f, where the weight of every vertex i is $w(i) = 2(n+1) - i$, following the convention $f(i) = i$.

Lemma 4.1 *An $(n-5)$-regular distance magic graph G with n vertices, where $n \equiv 5 \pmod{12}$, exists for all $n \geq 17$.*

Proof. Let $n = 12k + 5$ for $n \geq 17$, denote $t = 2k$. Let G be a 4-regular graph consisting of $k+1$ components: one component K_5 (see Figure 2) and k components $C_4[C_3]$, which is a composition of C_4 and C_3, see Figure 3.

Let $(a_{i,j})$ be a Kotzig array with 3 rows and $2t+1$ columns, let j be the column of the array containing $3t+2$. Let $u_1, u_2, \ldots, u_5$ be the vertices of K_5. We define auxiliary labeling f as follows.

$$f(u_1) = 1, \quad f(u_2) = 1 + a_{1,j}, \quad f(u_3) = 1 + a_{2,j}, \quad f(u_4) = 1 + a_{3,j}, \quad f(u_5) = 6t + 5$$

The weight of each vertex in K_5 is the sum of labels of all remaining vertices.

$$\begin{aligned}
w(u_1) &= 15t + 14 = 5(n+1)/2 - 1, \\
w(u_2) &= 15t + 14 - a_{1,j} = 5(n+1)/2 - f(u_2), \\
w(u_3) &= 15t + 14 - a_{2,j} = 5(n+1)/2 - f(u_3), \\
w(u_4) &= 15t + 14 - a_{3,j} = 5(n+1)/2 - f(u_4), \\
w(u_5) &= 9t + 10 = 5(n+1)/2 - f(u_5)
\end{aligned}$$

since $a_{1,j} + a_{2,j} + a_{3,j} = 9t + 6$. Notice, the weights follow (3).

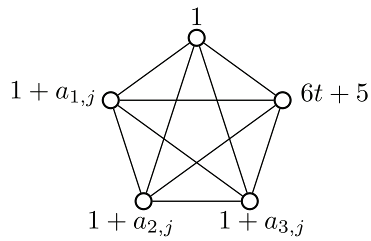

Figure 2: An auxiliary labeling of K_5.

Next, we extend labeling f to vertices of the k-th component $C_4[C_3]$. We take two pairs (x, y) and (w, z) of columns of the Kotzig array guaranteed by Lemma 3.1. We label the vertices in $C_4[C_3]$ as shown in Figure 3. Now, the weight of $1 + a_{1,x}$ (vertex labeled $1 + a_{1,x}$) is

$$\begin{aligned}
w(1 + a_{1,x}) &= ((1 + a_{2,x}) + (1 + a_{3,x})) + ((1 + a_{1,w}) + (1 + a_{3,z})) \\
&= (9t + 8 - a_{1,x}) + (6t + 6) = 15t + 14 - a_{1,x} = 5(n+1)/2 - f(a_{1,x}).
\end{aligned}$$

Again, the weight follows (3). In the same fashion we label all $C_3[C_4]$ components always with numbers from a different pair of columns from the Kotzig array. Thus, labeling f is an auxiliary labeling of G with $12k + 5$ vertices, where G has one component K_5 and k components $C_3[C_4]$.

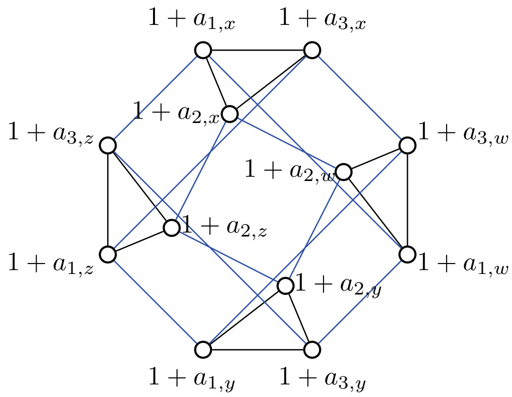

Figure 3: An auxiliary labeling of a component $C_4[C_3]$.

Complement $\overline{G}$ is a connected $12k$-regular graph with $12k+5$ vertices. Using the same labels for every vertex as in f, by (3) we obtain a labeling that has the same weight for every vertex. Clearly, the labeling is a 1-to-1 mapping to the set of $\{1, 2, \ldots, 12k+5\}$, thus f is a distance magic labeling of an $(n-5)$-regular graph with n vertices. □

A similar construction was found for $n \equiv 1, 3, 7, 9, 11 \pmod{12}$. Each construction uses heavily Lemma 3.1, just the first component differs for each case. We provide the full constructions in the journal paper published after the conference.

5 Conclusion

The main objective of this paper is to study the existence of regular distance magic graphs with an odd number of vertices. For graphs with an even number of vertices a single construction always works. The odd case requires several different approaches since many exceptional cases do not allow for a distance magic graph, unlike the even case.

A brute force computer search reveals, no $(n-5)$-regular distance magic graph with less than 15 vertices exists. An example of an $(n-5)$-regular distance magic graph with 15 vertices is in Figure 4.

Based on Lemma 4.1 and analogous lemmas for $n \equiv 1, 3, 7, 9, 11 \pmod{12}$ we coclude the following.

Theorem 5.1 *An $(n-5)$-regular distance magic graph G with n vertices exists for all odd feasible values $n \geq 15$.*

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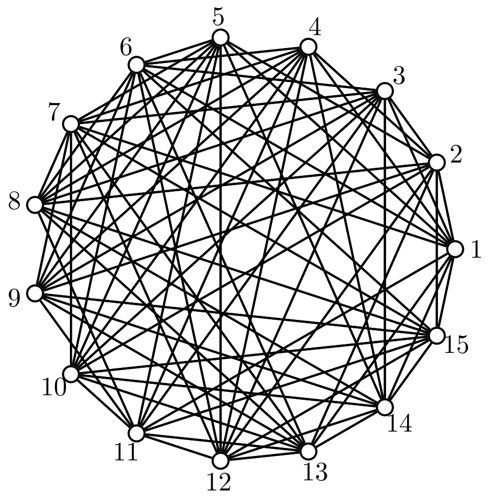

Figure 4: A distance magic 10-regular graph with 15 vertices.

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Husté distančně magické grafy

Abstrakt (Streszczenie): Mějme graf $G = (V, E)$ s n vrcholy. Bijektivní zobrazení f množiny V do $\{1, 2, \ldots, n\}$ se nazývá *distančně magické ohodnocení* grafu G, jestliže pro každý vrchol grafu G je součet ohodnocení sousedních vrcholů roven stejné hodnotě k. Graf, pro který takové ohodnocení existuje, nazýváme *distančně magický graf.* Pro grafy se sudým počtem vrcholů existuje jednoduchá konstrukce r-pravidelných distančně magických grafů pro všechny přípustné pravidelnosti r. Pro grafy s lichým počtem vrcholů jsou známy některé nutné podmínky existence a několik konstrukcí pro některé pravidelnosti. V tomto článku se věnujeme konstrukci distančně magických grafů s vysokou hodnotou r: konstruujeme $(n-5)$-pravidelné distančně magické grafy s n vrcholy. Magická ohodnocení se využívají při losování sportovních turnajů.
Klíčová slova (Słowa kluczowe): ohodnocení, distančně magický graf, pravidelný graf.