Numerical Approaches for Beams on Nonlinear Foundation - Part 1 (Theory)

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Abstract: Our work presents theory and numerical approaches suitable for the solutions of straight plane beams rested on elastic foundations (i.e. nonlinear modified bilateral and unilateral Winkler's models). The nonlinear boundary value problems of 4th-order are solved via finite element method with semi-smooth Newton's method (which discretize the weak formulation of the problem) and central difference method with classical Newton's method (which discretize directly the differential equation). Reaction forces in foundation are defined via nonlinear dependencies based on previous experiments.

Keywords: unilateral and bilateral elastic foundation, nonlinear foundation, beam, Finite Element Method, semi-smooth Newton's method, Central Difference Method.

1 Introduction

There are beams on elastic foundations which are frequently used in the engineering practice; for example see Fig. 1 and 2 and references [2], [3] and [4]. The first theory for the bending of beams on an elastic foundation was proposed by E. Winkler in the Prague in 1867; see [9]. The basic analysis of the bending of beams on an elastic foundation is based on the first assumption that the strains (i.e. deformations) are small.

Figure 1: *Examples of beams on elastic foundations (a) Railroads (b) Femoral screws in femur - rtg snapshot.*

In classical problems of engineering/mechanics, the deflection $v = v(x)$ [m] of the straight beam without any volume loads is described by linear/nonlinear differential equation

$$EJ_{ZT}\frac{d^4v}{dx^4} + q_R = 0,$$

where E [Pa] is the modulus of elasticity of the beam, $J_{ZT} = \int_A y^2\,\mathrm{d}A\,[\mathrm{m}^4]$ is the major principal second moment of the beam cross-section $A\,[\mathrm{m}^2]$ and $q_R = q_R(x, v, \ldots)\,[\mathrm{Nm}^{-1}]$ corresponds to the linear/nonlinear reaction of the foundation; see Fig. 2. The beam is loaded by force F [N].

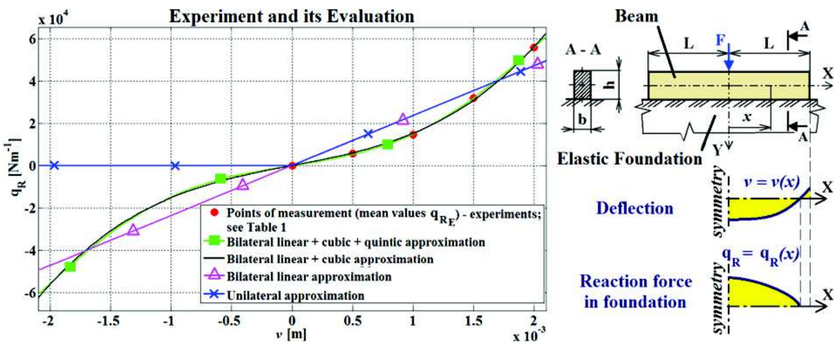

Figure 2: *(a) Dependence of reaction force on deflection (i.e. foundation load-settlement behaviour for a sand, experiment and suitable linear and nonlinear approximations) (b) Beam with cross-section $b \times h$ and length* 2L *is resting on elastic unilateral and bilateral foundation, see [2], [3], [6].*

Our work focuses on the numerical approaches for the solution of straight plane (2D) beams of length 2L on an elastic foundation with nonlinear unilateral or bilateral behaviour (linear Bernouli's beam, small deformations in the beam, Finite element Method, Central Difference Method); see Fig. 2. The methodology of the elastic foundation measuring applied in this paper is based on the pressing of a beam into the foundation; see Fig. 2, Tab. 1 and references [2], [3] and [5]. Hence, in this article, the theory and numerical nonlinear approaches are explained.

Table 1: Elastic foundation - Experiments and their evaluation

Description:	**Expression; see Fig. 2 (a)**	
Experiments	q_{R_E}	measurements (mean values)
Bilateral linear	$q_{R_1} = k_1 v = 2.3587 \times 10^7 v$	$\frac{d^4v}{dx^4} + \frac{k_1 v}{EJ_{ZT}} = 0$
Bilateral linear + cubic	$q_{R_{1,3}} = k_1 v + k_3 v^3 = 1.094 \times 10^7 v + 4.2869 \times 10^{12} v^3$	$\frac{d^4v}{dx^4} + \frac{k_1 v + k_3 v^3}{EJ_{ZT}} = 0$
Bilateral linear + cubic + quintic	$q_{R_{1,3,5}} = k_1 v + k_3 v^3 + k_5 v^5 = 8.8597 \times 10^6 v + 6.4373 \times 10^{12} v^3 - 4.1846 \times 10^{17} v^5$	$\frac{d^4v}{dx^4} + \frac{k_1 v + k_3 v^3 + k_5 v^5}{EJ_{ZT}} = 0$
Unilateral	$q_{\lvert R_1 \rvert} = \frac{k_1}{2}(v + \lvert v \rvert) = \frac{2.3587 \times 10^7}{2}(v + \lvert v \rvert)$	$\frac{d^4v}{dx^4} + \frac{k_1}{2EJ_{ZT}}(v + \lvert v \rvert) = 0$

2 Finite Element Method (FEM) Approach to Unilateral Elastic Foundation

Let us suppose that the solved beam has symmetry. Therefore it is sufficient to solve the differential equation for a half of the beam, i.e. $x \in \langle 0; \mathrm{L} \rangle$.

Hence, the deflection of the beam is described by the equation

$$EJ_{ZT} \frac{d^4 v}{dx^4} + kv^+ = 0 \qquad \text{on} \quad x \in (0, \mathrm{L})$$

with following boundary conditions prescribed in points $x = 0$ and $x = \mathrm{L}$

$$\begin{aligned} \frac{dv(x=0)}{dx} &= 0, \qquad & M_o(x = \mathrm{L}) &= 0, \\ T(x=0) &= \frac{\mathrm{F}}{2}, \qquad & T(x = \mathrm{L}) &= 0. \end{aligned} \tag{1}$$

2.1 Weak Formulation and FEM

Lets denote V as the space of virtual displacements then $V = \left\{ w \in H^2\left((0, \mathrm{L})\right) : \frac{dw(x=0)}{dx} = 0 \right\}$.

The weak formulation of the beam defection on the unilateral foundation is following

$$\begin{aligned} &\text{find } v \in V \text{ such that} \\ &EJ_{ZT} \int_0^{\mathrm{L}} \frac{d^2 v}{dx^2} \frac{d^2 w}{dx^2} \, dx + k \int_0^{\mathrm{L}} v^+ w \, dx = \frac{\mathrm{F}}{2} w(0) \text{ is fulfilled for all } w \in V. \end{aligned} \tag{2}$$

For the sake of solvability of (2) the prescribed external force F must be positive. See (Sysala 2008) for details.

Lets divide the interval $(0, \mathrm{L})$ into n parts of the same length. This equidistant discretization with nodes $x_1 = 0, x_{i+1} = x_i + h$ has the constant step $h = \mathrm{L}/n$.

The discrete approximation of the space V is subspace V_h such that

$$V_h = \left\{ v_h \in C^1\big((0, \mathrm{L})\big) : \; v_h|_{\langle x_i, x_{i+1} \rangle} \in P_3, \; \frac{dv_h(0)}{dx} = 0 \right\}.$$

The discrete form of (2) is following

$$\begin{aligned} &\text{find } v_h \in V_h \text{ such that} \\ &EJ_{ZT} \int_0^{\mathrm{L}} \frac{d^2 v_h}{dx^2} \frac{d^2 \varphi_i}{dx^2} \, dx + k \int_0^{\mathrm{L}} v_h^+ \varphi_i \, dx = \frac{\mathrm{F}}{2} \varphi_i(0) \text{ for all } i = 1, \ldots, 2n+2, \end{aligned} \tag{3}$$

where φ_i, $i = 1, \ldots 2n+2$ are piecewise-cubic smooth functions, the base function of space V_h. Because the solution v_h of (3) is element of the space V_h, we can write

$$v_h = \sum_{i=1}^{2n+2} u_i \varphi_i(x). \tag{4}$$

And we will denote the vector $\boldsymbol{u}$

$$\boldsymbol{u} = \Big(\underbrace{v_h(x_1)}_{u_1}, \underbrace{\frac{dv_h(x_1)}{dx}}_{u_2=0}, \underbrace{v_h(x_2)}_{u_3}, \underbrace{\frac{dv_h(x_2)}{dx}}_{u_4}, \ldots, \underbrace{v_h(x_{n+1})}_{u_{2n+1}}, \underbrace{\frac{dv_h(x_{n+1})}{dx}}_{u_{2n+2}} \Big)^\top.$$

The algebraic FEM representation of the first integral in (3) and the right side of (3) can be set by a standart way. The global stiffness matrix $\boldsymbol{K}$ and the global load vector $\boldsymbol{f}$ corresponding to (3) are shown. ($h = \mathrm{L}/n$ constant).

$$\boldsymbol{K} = \frac{1}{h^3}\begin{pmatrix} 12 & 0 & -12 & 6h & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & h^3 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -12 & 0 & 24 & 0 & -12 & 6h & \dots & 0 & 0 & 0 & 0 \\ 6h & 0 & 0 & 8h^2 & -6h & 2h^2 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6h & 24 & 0 & \ddots & -12 & 6h & 0 & 0 \\ 0 & 0 & 6h & 2h^2 & 0 & 8h^2 & \ddots & -6h & 2h^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6h & \ddots & 24 & 0 & -12 & 6h \\ 0 & 0 & 0 & 0 & 6h & 2h^2 & \ddots & 0 & 8h^2 & -6h & 2h^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -12 & -6h & 12 & -6h \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 6h & 2h^2 & -6h & 4h^2 \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} \frac{\mathrm{F}}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.2 Semi-Smooth Newton's Method and Numerical Algorithm

We will present here one way how to find a numerical algorithm to the problem (3), which was present in [7] for the case of the thin annular plate. Notice, there is the nonlinear expression

$$v_h^+ = (v_h(x))^+ \;=\; \frac{1}{2}(|v_h(x)| + v_h(x)) \;=\; \frac{1}{2}\left(\left|\sum_{i=1}^{2n+2} u_i\varphi_i(x)\right| + \sum_{i=1}^{2n+2} u_i\varphi_i(x)\right)$$

in the second integral on the left side of the equation. We deal with it in two steps.

The 1$^{\text{st}}$ step. We use the well-known trapezoidal rule[1] for approximation of the integral $\int_0^{\mathrm{L}} v_h^+\varphi_i\,dx$ from the left side of the equation in (3). The main reason is that we get the following approximation

$$\int_0^{\mathrm{L}} v_h^+\varphi_i\,dx \approx \begin{cases} \frac{1}{2}\,h\,u_1^+\,, & \text{if } i = 1\,, \\ h\,u_i^+\,, & \text{if } i \text{ is odd, } i \neq 1, i \neq 2n+1\,, \\ \frac{1}{2}\,h\,u_{2n+1}^+\,, & \text{if } i = 2n+1\,, \\ 0\,, & \text{if } i \text{ is even,} \end{cases}$$

which moves the non-linearity $(\,\cdot\,)^+$ from the function v_h from (4) to its finite element components $u_i \in \mathbb{R}$ and therefore the evaluation is easy in any numerical algorithm. Now we get the homogenous equation

$$G(u) = 0 \tag{5}$$

instead of (3) for the non-linear mapping

$$G(u) = EJ_{ZT}\boldsymbol{K}\boldsymbol{u} + k\boldsymbol{B}\boldsymbol{u}^+ - \boldsymbol{f}\,, \tag{6}$$

where the matrix $\boldsymbol{K}$ and the vector $\boldsymbol{f}$ are from the finite element method mentioned above and the matrix $\boldsymbol{B}$ is diagonal, $\boldsymbol{B} = \mathrm{diag}(h/2, 0, h, 0, h, 0\ \dots\ , h, 0, h/2, 0)$.

[1]Exactly, we use the trapezoidal rule with the same grid as is used in the finite element method, see above.

The 2nd step. Because we do not have available any derivative of (6) due to the absolute value in $\boldsymbol{u}^+$, . For this reason, the second step to find the numerical algorithm is the usage of more suitable semi-smooth Newton's method, see [1], which introduces so called slanting function G^o and use it instead of Jacobian in the standard Newton's iterations. We define

$$\boldsymbol{G}^o(u) = EJ_{ZT}\boldsymbol{K} + k\boldsymbol{B}\text{diag}(A(\boldsymbol{u}^+))$$

in our case, where the symbol $(A(\boldsymbol{u}^+))$ stands for the active set of indexes of such nods x_i, in which the elastic beam foundation is active. The resulting iterative equation in the $(k+1)$-th step of the semi-smooth Newton's method is

$$\boldsymbol{u}^{(k+1)} = \boldsymbol{u}^{(k)} - \boldsymbol{G}^o\left(\boldsymbol{u}^{(k)}\right)^{-1} G(\boldsymbol{u}^{(k)})$$

for known solution $\boldsymbol{u}^{(k)}$ from the previous step. This iteration process converges for sufficiently small distance between the initial estimate $\boldsymbol{u}^{(0)}$ and the exact solution of the equation (5), for details see [1].

There must be defined the suitable starting estimate $\boldsymbol{u}^{(0)}$ in our computational algorithm. We use the result deflection of the beam without any foundation. This deflection is solution of equation (3) without the part with v_h^+.

The computational process of the algorithm is repeated until at least one termination condition has been reached:

- either the solution $\boldsymbol{u}^{(k+1)}$ satisfies the criteria of sufficiently small relative error

$$\frac{\|\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)}\|}{\|\boldsymbol{u}^{(k)}\|}$$

- or the "exact" solution is reached at which the relative reziduum

$$\frac{\|\boldsymbol{G}^o(\boldsymbol{u}^{(k+1)})\,\boldsymbol{u}^{(k+1)} - \boldsymbol{f}\|}{\left(EJ_{ZT}\,\|\boldsymbol{K}\| + k\,\|\boldsymbol{B}\text{diag}\left(A\left((\boldsymbol{u}^{k+1})^+\right)\right)\|\right)\|\boldsymbol{u}^{(k+1)}\| + \|\boldsymbol{f}\|}$$

  vanishes.

3 Central Difference Method (CDM) Approach to Bilateral Elastic Foundation

According to the theory of CDM, the beam and its surroundings can be divided into $n+5$ nodes "i" with step $\Delta = \frac{L}{n}$; see Fig. 3.

Central differences (CD) at the point "i" can be defined as an approximation of derivatives $v^{(i)} = \frac{d^i v}{dx^i}$. Hence,

$$v^{(1)} \approx \frac{v_{i+1} - v_{i-1}}{2\Delta}, \qquad v^{(2)} \approx \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2},$$

$$v^{(3)} \approx \frac{v_{i+2} - 2v_{i+1} + 2v_{i-1} - v_{i-2}}{2\Delta^3}, \qquad v^{(4)} \approx \frac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{\Delta^4}$$

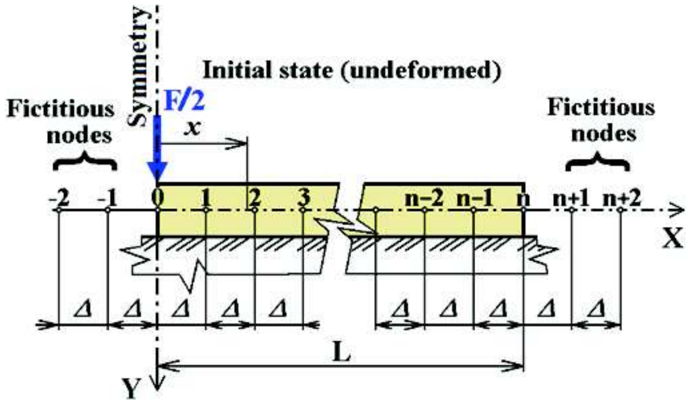

Figure 3: *Beam of length* 2L *resting on an elastic foundation and loaded by force* F *and divisions of the beam (CDM – one half of the beam).*

For example, differential equation

$$EJ_{ZT}\frac{d^4v}{dx^4} + k_1v + k_3v^3 + k_5v^5 = 0$$

can be approximated via CD as

$$v_{i-2} - 4v_{i-1} + (6 + a_1)v_i - 4v_{i+1} + v_{i+2} + a_3v_i^3 + a_5v_i^5 = 0 \quad \text{for } i = 0, 1, 2, \ldots, n,$$

where $a_1 = \frac{k_1\Delta^4}{EJ_{ZT}}$, $a_3 = \frac{k_3\Delta^4}{EJ_{ZT}}$, $a_5 = \frac{k_5\Delta^4}{EJ_{ZT}}$, $c = 6 + a_1$ and $b = \frac{\mathrm{F}\Delta^3}{EJ_{ZT}}$. The variables v_{-2}, v_{-1}, v_{n+1} and v_{n+2} (i.e. connection with fictitious nodes) can be expressed from boundary conditions. Hence, $v_{-1} = v_1$, $v_{-2} = v_2 - b$, $v_{n+1} = 2v_n - v_{n-1}$ and $v_{n+2} = 4v_n - 4v_{n-1} + v_{n-2}$. For more information see [2], [3], [8] and [6] . This leads to a system of $n + 1$ nonlinear equations

$$\boldsymbol{Mv} + a_3\boldsymbol{v}^3 + a_5\boldsymbol{v}^5 - \boldsymbol{b} = \boldsymbol{o}, \tag{7}$$

where

$$\boldsymbol{M} = \begin{pmatrix} c & -8 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -4 & 7+a_1 & -4 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -4 & c & -4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -4 & c & -4 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 1 & -4 & c & -4 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -4 & c & -4 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -4 & 5+a_1 & -2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 2 & -4 & 2+a_1 \end{pmatrix},$$

$$\boldsymbol{b} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \boldsymbol{v}^3 = \begin{pmatrix} v_0^3 \\ v_1^3 \\ \vdots \\ v_n^3 \end{pmatrix}, \quad \boldsymbol{v}^5 = \begin{pmatrix} v_0^5 \\ v_1^5 \\ \vdots \\ v_n^5 \end{pmatrix}.$$

The nonlinear equation (7) is solved by well-known Newton's method (also known as the Newton–Raphson method) with the following termination condition

$$\|\boldsymbol{v}^{(k+1)} - \boldsymbol{v}^{(k)}\| < \varepsilon_{\text{tol}}\,,$$

which means that the distance of two last iteration solutions $\boldsymbol{v}^{(k+1)}$ and $\boldsymbol{v}^{(k)}$ is sufficiently small.

Conclusion

There is considered the boundary value problems describing the deflection of the straight beam rested on two classes of nonlinear elastic foundations in our paper. The definitions of the reaction forces in the foundations are based on previous experiments described in the previous papers listed in the references. The first class is case of beam on unilateral foundation. We have described the derivation of the finite element method formulation of the problem and then we have suggested the computational algorithm via semi-smooth Newton's method. On the other hand we have used the central difference method in the case of bilateral foundations, which is the second class of nonlinear elastic foundations. And we use the classical Newton's method to solve the resulting equation. Both approaches lead to computational algorithms through which we are able to get the numerical solutions which are comparable with analytical solutions with good results.

Acknowledgement

The authors gratefully acknowledge the funding from the Czech projects SP2016/145.

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Numerické řešení ohybu nosníku v nelineárním prostředí - část 1 (teorie)

Abstrakt: V článku se zabýváme numerickým řešením úlohy, která popisuje ohyb rovinného nosníku uloženého v různých typech elastického prostředí (t.j. nelineární modifikované bilaterální a unilaterální Winklerova typu). Jsou popsány dva způsoby řešení okrajové úlohy s nelineární diferenciální rovnicí čtvrtého řádu. První je pomocí metody konečných prvků s využitím nehladké Newtonovy metody a druhý je založený na metodě centrálních diferencí a použití klasické Newtonovy metody. Reakční síly v podloží jsou definovány nelineárními zobrazeními, jejichž tvar vychází z předchozích experimentů.

Klíčová slova: jednostranné a oboustranné elastické podloží, nelineární podloží, nosník, metoda konečných prvků, nehladká Newtonova metoda, metoda konečných diferencí.