Globally variational forms on the Möbius strip: Examples

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Abstract: Examples of globally variational source forms (differential equations) defined on the open Möbius strip are studied by means of the Vainberg–Tonti construction. Our choice of the underlying space follows from the variational sequence theory on fibered manifolds, which guarantees global variationality over topological spaces with trivial the second de Rham cohomology group.

Keywords: Möbius strip, Helmholtz conditions, Vainberg–Tonti Lagrangian, variational sequence, global variationality.

1 Introduction

In this paper we study simple examples of variational differential forms on the open Möbius strip, a representative of smooth manifolds possessing *trivial* the second de Rham cohomology group. This topological property of the underlying space assures that locally variational forms are automatically *globally variational,* which is the important result of the *variational sequence* theory over fibered manifolds, the main tool for study the local and global properties of the Euler-Lagrange mapping in the calculus of variations (cf. Krupka [5,7], Takens [11], Urban and Krupka [14], Volná and Urban [16]). Although the existence of a global variational principle is guaranteed by the theory, there is no general construction of a *global Lagrangian* for given differential equations (source forms), defined on this class of underlying manifolds. In the concrete examples, we applied the Vainberg–Tonti construction and obtained the corresponding globally defined Lagrange functions. Nevertheless, the general theory requires further research.

Basic concepts of the geometric theory of second-order variational differential equations are recalled in a slightly simplified setting. For the general theory of global variational principles on fibered manifolds we refer to Krupka [7], and references therein; see also Anderson and Duchamp [1], Brajerčík and Krupka [2], Krupka, Urban, and Volná [8], Krupková and Prince [9].

Throughout, Y denotes a fibered manifold with base X and projection π. The r-jet prolongation of Y is denoted by J^rY, and $\pi^r : J^rY \to X$, $\pi^{r,0} : J^rY \to Y$ are the canonical jet projections. For an open subset $W \subset Y$, we put $W^r = (\pi^{r,0})^{-1}(W)$. The ring of functions on W^r is denoted by $\Omega_0^r W$, and the $\Omega_0^r W$-module of differential k-forms on W^r is denoted by $\Omega_k^r W$. $M_{r,a}$ denotes the open Möbius strip (that is, without boundary) of radius r and width $2a$, where $0 < a < r$.

2 Variational equations and the Vainberg–Tonti Lagrangian

Let W be an open subset of a fibered manifold Y over 1-dimensional base X ("fibered mechanics"). Consider a *source form* $\varepsilon \in \Omega^2_{2,Y} W$ (also called a *dynamical form* in Lagrangian mechanics), which is by definition a 1-contact, $\pi^{2,0}$-horizontal 2-form, defined on an open subset $W^2 \subset J^2 Y$. In a fibered chart (V, ψ), $\psi = (t, x^i)$, ε is expressed by

$$\varepsilon = \varepsilon_i \omega^i \wedge dt, \tag{2.1}$$

where

$$\omega^i = dx^i - \dot{x}^i dt \tag{2.2}$$

are *contact* 1-forms on V^1, and the coefficients $\varepsilon_i = \varepsilon_i(t, x^j, \dot{x}^j, \ddot{x}^j)$ are real-valued functions on V^2. Every *Lagrangian* $\lambda \in \Omega^1_{1,X} W$, by definition a π^1-horizontal 1-form on $W^1 \subset J^1 Y$, induces a source form E_λ, expressed in a fibered chart (V, ψ), $\psi = (t, x^i)$, by

$$E_\lambda = \mathscr{E}_i(L)\omega^i \wedge dt,$$

where $\lambda = L dt$, and

$$\mathscr{E}_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial t \partial \dot{x}^i} - \frac{\partial^2 L}{\partial x^j \partial \dot{x}^i}\dot{x}^j - \frac{\partial^2 L}{\partial \dot{x}^j \partial \dot{x}^i}\ddot{x}^j$$

are the *Euler–Lagrange expressions* associated to L.

ε is called *locally variational*, if there is a family of Lagrangians $(\lambda_\iota)_{\iota \in I}$, $\lambda_\iota \in \Omega^1_{1,X} V_\iota$, defined on an open covering $(V_\iota)_{\iota \in I}$ of Y such that

$$\varepsilon|_{V_\iota} = E_{\lambda_\iota}. \tag{2.3}$$

In a fibered chart (V, ψ), $\psi = (t, x^i)$, a Lagrangian has an expression $\lambda = L dt$, and condition (2.3) means that the coefficients ε_i of ε coincide with the Euler–Lagrange expressions of a Lagrange function $L = L(t, x^j, \dot{x}^j)$, that is

$$\varepsilon_i = \mathscr{E}_i(L).$$

ε is called *globally variational* (or simply *variational*), if there exists a Lagrangian $\lambda \in \Omega^1_{1,X} W$ such that $\varepsilon = E_\lambda$.

Remark 1. Clearly, this concept of (local) variationality transfers to systems of m second-order ordinary differential equations. In a chart (V, ψ), $\psi = (t, x^i)$, we have a system

$$\varepsilon_i(t, x^j, \dot{x}^j, \ddot{x}^j) = 0, \tag{2.4}$$

where $i, j = 1, 2, \dots, m$ (the number of equations and dependent variables are equal). Solutions of (2.4) are differentiable mappings γ defined on an open interval in $\mathbb{R}$ with values in $\mathbb{R}^m$, $\zeta(t) = (x^1\zeta(t), x^2\zeta(t), \dots, x^m\zeta(t))$, which satisfy (2.4). System (2.4) is called *locally variational*, if (2.4) coincides with the Euler–Lagrange equations for some Lagrange functions $L = L(t, x^j, \dot{x}^j)$.

Remark 2. For the purpose of this paper, we shall work with a Cartesian product $Y = \mathbb{R} \times M$ fibered over $\mathbb{R}$ (endowed with a canonical global coordinate), where M is a submanifold of a Euclidean space, the *Möbius strip.* In case of this trivial fibration, the notion of globally variational source form reduces to existence of a globally defined Lagrange function on the corresponding underlying set.

The following theorem describes necessary and sufficient conditions for ε (2.1) to be locally variational.

Theorem 3 (Helmholtz conditions)**.** *Let* (V, ψ)*,* $\psi = (t, x^i)$*, be a fibered chart on* $W \subset Y$*, and* $\varepsilon \in \Omega^2_{2,Y} W$ *be a source form with the expression* (2.1)*. The following two conditions are equivalent:*

(a) ε *is locally variational.*

(b) *The functions* ε_i *satisfy the system*

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = 0, \tag{2.5}$$

$$\frac{\partial \varepsilon_i}{\partial \dot{x}^j} + \frac{\partial \varepsilon_j}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i}\right) = 0, \tag{2.6}$$

$$\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i}\right) = 0. \tag{2.7}$$

Proof. The Helmholtz conditions (2.5)–(2.7) were obtained by von Helmholtz [13]; for the proof see e.g. Havas [3]. Generalized conditions for higher-order partial differential equations can be found in Krupka [4]. □

Remark 4. It is straightforward that conditions (2.5) and (2.6) imply linearity of ε_i in the second derivatives, i.e. $\varepsilon_i = A_i + B_{ij}\ddot{x}^j$, and the property $B_{ij} = \partial C_i / \partial \dot{x}^j = \partial C_j / \partial \dot{x}^i = B_{ji}$ for some functions $C_i = C_i\left(t, x^j, \dot{x}^j\right)$. Hence the Helmholtz conditions (2.5)–(2.7) for ε_i can be equivalently reformulated for first-order functions A_i, B_{ij} (cf. Sarlet [10]).

Another standard result is a construction of a Lagrangian for locally variational source form.

Theorem 5 (Vainberg–Tonti)**.** *Let* (V, ψ)*,* $\psi = (t, x^i)$*, be a fibered chart on* $W \subset Y$ *such that* $\psi(V)$ *is star-shaped, and* $\varepsilon \in \Omega^2_{2,Y} W$ *be a source form with the expression* (2.1)*. If* ε *is locally variational, then* $\varepsilon|_V = E_\lambda$*, where* $\lambda \in \Omega^2_{1,X} V$*,* $\lambda = L dt$*, and*

$$L\left(t, x^i, \dot{x}^i, \ddot{x}^i\right) = x^i \int_0^1 \varepsilon_i\left(t, s x^i, s\dot{x}^i, s\ddot{x}^i\right) ds. \tag{2.8}$$

Proof. We refer to Tonti [12]; see also Krupka [6]. □

Remark 6. Note that in the context of Theorem 5, the *Vainberg–Tonti Lagrangian* $\lambda \in \Omega^2_{1,X} V$, given by (2.8), can always be reduced to a *first-order* Lagrangian by means of deleting some total derivative terms.

3 Variational sequence theory in fibered mechanics

We now very briefly recall a sheaf-theoretic concept in the variational calculus on fibered manifolds, the variational sequence theory and its consequences for global variationality; our main reference is Krupka [5], see also Urban and Krupka [14]. The construction can be described rather simply: the de Rham sequence of differential forms on the corresponding underlying manifold is factored through its contact subsequence. It turns out, in particular, that one of the quotient morphisms coincides with the Euler-Lagrange mapping, assigning to a Lagrangian its Euler–Lagrange form. The quotient sheaf sequence, the *variational sequence*, then can be used to study the local and global properties of the Euler-Lagrange mapping. We have the commutative diagram

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & 0 & & \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & & \nearrow & \Omega^r_1/\Theta^r_1 & \longrightarrow & \Omega^r_2/\Theta^r_2 & \longrightarrow & \Omega^r_3/\Theta^r_3 & \longrightarrow & \cdots \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^r_0 & \longrightarrow & \Omega^r_1 & \longrightarrow & \Omega^r_2 & \longrightarrow & \Omega^r_3 & \longrightarrow & \cdots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & 0 & \longrightarrow & \Theta^r_1 & \longrightarrow & \Theta^r_2 & \longrightarrow & \Theta^r_3 & \longrightarrow & \cdots \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & & & 0 & & 0 & & 0 & &
\end{array}
$$

Theorem 7. *The variational sequence of order r over Y is an acyclic resolution of the constant sheaf $\mathbb{R}_Y$ over Y.*

Proof. See Krupka [5]. □

From Theorem 7 and the well-known Abstract de Rham theorem (cf. Wells [17]), we get the next result.

Corollary 8. *The cohomology of the complex of global sections of the variational sequence and the de Rham cohomology of Y coincide,*

$$H^k\left(\Gamma\mathcal{V}^r Y\right) = H^k_{deR}Y, \quad k \geq 0.$$

The following assertion follows from Corollary 8 and properties of the variational sequence.

Corollary 9. *Suppose ε be a source form on J^rY. If ε is locally variational and the de Rham cohomology group $H^2_{deR}Y$ is trivial, then ε is globally variational.*

4 Smooth atlas adapted to fibered Möbius strip

For the main purpose of this work, the study of examples of globally variational forms on the Möbius strip, we give its smooth manifold structure. Consider the open subset

$$W = \mathbb{R} \times \{\mathbb{R}^3 \backslash \{(0, 0, z)\}\}$$

in the Euclidean space $\mathbb{R}^4$, endowed with its open submanifold structure. The global Cartesian coordinates on W are denoted by (t, x, y, z). We introduce an atlas on W adapted to the fibered Möbius strip $\mathbb{R} \times M_{r,a}$ as follows. Let V and $\bar{V}$ be an open covering of W, where

$$V = \mathbb{R} \times \mathbb{R}^3 \backslash ((-\infty, 0] \times \{0\} \times \mathbb{R}), \quad \bar{V} = \mathbb{R} \times \mathbb{R}^3 \backslash ([0, \infty) \times \{0\} \times \mathbb{R}),$$

and define coordinate functions $(t, \varphi, \tau, \kappa)$ on V by $t = t$,

$$\begin{aligned}
\varphi &= \operatorname{atan2}(y, x), \\
\tau &= \frac{1}{\sqrt{2}} \left(\sqrt{x^2 + y^2} - r \right) \sqrt{1 + \frac{x}{\sqrt{x^2 + y^2}}} + \frac{1}{\sqrt{2}} \operatorname{sgn}(y)\, z \sqrt{1 - \frac{x}{\sqrt{x^2 + y^2}}}, \\
\kappa &= -\frac{1}{\sqrt{2}} \left(\sqrt{x^2 + y^2} - r \right) \operatorname{sgn}(y) \sqrt{1 - \frac{x}{\sqrt{x^2 + y^2}}} + \frac{1}{\sqrt{2}} z \sqrt{1 + \frac{x}{\sqrt{x^2 + y^2}}},
\end{aligned}$$

and $(\bar{t}, \bar{\varphi}, \bar{\tau}, \bar{\kappa})$ on $\bar{V}$ by $\bar{t} = t$, $\bar{\tau} = -\tau$, $\bar{\kappa} = -\kappa$, and

$$\bar{\varphi} = \begin{cases} \operatorname{atan2}(y, x), & y \geq 0, \\ \operatorname{atan2}(y, x) + 2\pi, & y < 0, \end{cases}$$

where $\operatorname{atan2}(y, x)$ is the arctangent function with two arguments.

It is easy to check that the pairs (V, Ψ), $\Psi = (t, \varphi, \tau, \kappa)$, and $(\bar{V}, \bar{\Psi})$, $\bar{\Psi} = (\bar{t}, \bar{\varphi}, \bar{\tau}, \bar{\kappa})$, are charts on W *adapted* to $\mathbb{R} \times M_{r,a}$, which form a smooth atlas on W (see Urban and Volná [15]). On the intersection $V \cap \bar{V} = \mathbb{R} \times \mathbb{R}^3 \setminus (\mathbb{R} \times \{0\} \times \mathbb{R})$, the chart transformations between (V, Ψ) and $(\bar{V}, \bar{\Psi})$ are expressed by

$$\Psi \circ \bar{\Psi}^{-1} : \bar{\Psi}(\bar{V}) \backslash \{\bar{\varphi} = \pi\} \to \Psi(V) \backslash \{\varphi = 0\},$$

$$\Psi \circ \bar{\Psi}^{-1}(\bar{t}, \bar{\varphi}, \bar{\tau}, \bar{\kappa}) = \begin{cases} (\bar{t}, \bar{\varphi}, \bar{\tau}, \bar{\kappa}), & \bar{\varphi} \in (0, \pi), \\ (\bar{t}, \bar{\varphi} - 2\pi, -\bar{\tau}, -\bar{\kappa}), & \bar{\varphi} \in (\pi, 2\pi), \end{cases} \tag{4.1}$$

and

$$\bar{\Psi} \circ \Psi^{-1} : \Psi(V) \backslash \{\varphi = 0\} \to \bar{\Psi}(\bar{V}) \backslash \{\bar{\varphi} = \pi\},$$

$$\bar{\Psi} \circ \Psi^{-1}(t, \varphi, \tau, \kappa) = \begin{cases} (t, \varphi, \tau, \kappa), & \varphi \in (0, \pi), \\ (t, \varphi + 2\pi, -\tau, -\kappa), & \varphi \in (-\pi, 0). \end{cases} \tag{4.2}$$

Note that in the chart (V, Ψ) (resp. $(\bar{V}, \bar{\Psi})$), $\mathbb{R} \times M_{r,a}$ has the equation $\kappa = 0$ with $-a < \tau < a$ (resp. $\bar{\kappa} = 0$ with $-a < \bar{\tau} < a$). The associated smooth atlas on $\mathbb{R} \times M_{r,a}$ is defined by the charts $(V \cap (\mathbb{R} \times M_{r,a}), \Psi|_{V \cap (\mathbb{R} \times M_{r,a})})$ and $(\bar{V} \cap (\mathbb{R} \times M_{r,a}), \bar{\Psi}|_{\bar{V} \cap (\mathbb{R} \times M_{r,a})})$, and we denote the associated coordinates by the same letters as $\Psi = (t, \varphi, \tau)$ and $\bar{\Psi} = (\bar{t}, \bar{\varphi}, \bar{\tau})$, if no misunderstanding may arise.

5 Globally variational forms: Examples

We now apply the variational sequence theory over fibered manifold $Y = \mathbb{R} \times M_{r,a}$. Since

$$H^2_{deR} M_{r,a} = 0, \tag{5.1}$$

Corollary 9 implies that every locally variational source form ε on $J^2(\mathbb{R} \times M_{r,a})$ is also *globally* variational. In order words, condition (5.1) assures existence of a Lagrange function defined on $J^1(\mathbb{R} \times M_{r,a})$, for which the corresponding Euler–Lagrange expressions coincide with ε (cf. Remark 2). We give examples of source forms on $J^2(\mathbb{R} \times M_{r,a})$, illustrating this sheaf-theoretic result.

5.1 The kinetic Lagrangian

Consider the canonical embedding $\iota : \mathbb{R} \times M_{r,a} \to \mathbb{R} \times \mathbb{R}^3$ and its jet prolongations $J^r\iota : J^r(\mathbb{R} \times M_{r,a}) \to J^r(\mathbb{R} \times \mathbb{R}^3)$. Denote by (t, x, y, z) the canonical coordinates on $\mathbb{R} \times \mathbb{R}^3$, and by $(t, x, y, z, \dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z})$ the associated coordinates on $J^2(\mathbb{R} \times \mathbb{R}^3)$. The source form

$$\varepsilon = \varepsilon_x \omega^x \wedge dt + \varepsilon_y \omega^y \wedge dt + \varepsilon_z \omega^z \wedge dt,$$

where ω^x, ω^y, and ω^z are contact 1-forms (2.2), and $\varepsilon_x = -\ddot{x}$, $\varepsilon_y = -\ddot{y}$, $\varepsilon_z = -\ddot{z}$, is variational and possesses a global Lagrangian, the *kinetic energy Lagrangian* $\lambda = L_{kin}dt$ on $J^1(\mathbb{R} \times \mathbb{R}^3)$, where

$$L_{kin} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \tag{5.2}$$

The induced source form $J^2\iota^*\varepsilon$ on $J^2(\mathbb{R} \times M_{r,a})$ is globally variational. Indeed, if $J^2\iota^*\varepsilon$ is expressed in the chart (V, Ψ), $\Psi = (t, \varphi, \tau, \kappa)$, as introduced in Sec. 4, we have $J^2\iota^*\varepsilon|_V = \varepsilon_\varphi \omega^\varphi \wedge dt + \varepsilon_\tau \omega^\tau \wedge dt$, where

$$\begin{aligned}
\varepsilon_\varphi = {} & \frac{1}{2}\dot{\varphi}^2 \tau \sin\frac{\varphi}{2}\left(r + \tau\cos\frac{\varphi}{2}\right) - \frac{1}{2}\dot{\varphi}\dot{\tau}\left(4\cos\frac{\varphi}{2}\left(r + \tau\cos\frac{\varphi}{2}\right) + \tau\right) \\
& - \left(\left(r + \tau\cos\frac{\varphi}{2}\right)^2 + \frac{\tau^2}{4}\right)\ddot{\varphi}, \\
\varepsilon_\tau = {} & \frac{1}{4}\dot{\varphi}^2\left(4\cos\frac{\varphi}{2}\left(r + \tau\cos\frac{\varphi}{2}\right) + \tau\right) - \ddot{\tau}.
\end{aligned}$$

The Vainberg–Tonti Lagrangian (2.8) associated with $J^2\iota^*\varepsilon|_V$ is of second order, and it can be reduced to the first-order Lagrangian, which coincides with $J^1\iota^*\lambda = (L_{kin} \circ J^1\iota)\,dt$ on $J^1(\mathbb{R} \times M_{r,a})$, where

$$L_{kin} \circ J^1\iota\,(\varphi, \tau, \dot{\varphi}, \dot{\tau}) = \frac{1}{2}\left(\dot{\tau}^2 + \left(\left(r + \tau\cos\frac{\varphi}{2}\right)^2 + \frac{\tau^2}{4}\right)\dot{\varphi}^2\right). \tag{5.3}$$

Using the chart transformations (4.1), (4.2), it is also easy to verify that formula (5.3) defines a global function on $J^1(\mathbb{R} \times M_{r,a})$.

5.2 The Vainberg–Tonti Lagrangian need not be global

We give another simple example of a globally defined source form ε on $J^1(\mathbb{R} \times M_{r,a})$, which is locally hence also globally variational. But contrary to the previous example, it shows that the direct use of the Vainberg–Tonti construction does *not* lead to the global Lagrangian.

Let ε be a source form defined on $J^1(\mathbb{R} \times M_{r,a})$ such that

$$\varepsilon|_V = \varepsilon_\varphi \eta^\varphi \wedge dt + \varepsilon_\tau \eta^\tau \wedge dt, \quad \varepsilon|_{\bar{V}} = \varepsilon_{\bar{\varphi}} \eta^{\bar{\varphi}} \wedge d\bar{t} + \varepsilon_{\bar{\tau}} \eta^{\bar{\tau}} \wedge d\bar{t},$$

where $\varepsilon_\varphi = 1 = \varepsilon_{\bar{\varphi}}$, $\varepsilon_\tau = 0 = \varepsilon_{\bar{\tau}}$, and $\eta^\varphi = d\varphi - \dot{\varphi}dt$, $\eta^\tau = d\tau - \dot{\tau}dt$, $\eta^{\bar{\varphi}} = d\bar{\varphi} - \dot{\bar{\varphi}}dt$, $\eta^{\bar{\tau}} = d\bar{\tau} - \dot{\bar{\tau}}dt$. Clearly, ε is locally variational (cf. Theorem 3). The Vainberg–Tonti Lagrangian

(2.8) associated with $\varepsilon|_V$, resp. $\varepsilon|_{\bar{V}}$, reads $L = \varphi$, $\varphi \in (-\pi, \pi)$, resp. $\bar{L} = \bar{\varphi}$, $\bar{\varphi} \in (0, 2\pi)$. These local Lagrange functions, however, do *not* define a global Lagrange function for given ε. Nevertheless, ε is globally variational, and it possesses a global Lagrange function, defined by

$$\mathscr{L}(t, \varphi, \dot{\varphi}) = \begin{cases} \varphi + \pi(1 + \cos\varphi) - \pi t \dot{\varphi} \sin\varphi, & \varphi \in (0, \pi), \\ \varphi + 2\pi, & \varphi \in (-\pi, 0], \end{cases}$$

and

$$\bar{\mathscr{L}}(\bar{t}, \bar{\varphi}, \dot{\bar{\varphi}}) = \begin{cases} \bar{\varphi} + \pi(1 + \cos\bar{\varphi}) - \pi \bar{t} \dot{\bar{\varphi}} \sin\bar{\varphi}, & \bar{\varphi} \in (0, \pi), \\ \bar{\varphi}, & \bar{\varphi} \in [\pi, 2\pi). \end{cases}$$

Clearly, $\bar{\mathscr{L}} \circ \bar{\Psi} \circ \Psi^{-1} = \mathscr{L}$ on $V \cap \bar{V}$.

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Globálně variační formy na Möbiově pásce: Příklady

Abstrakt: S využitím Vainberg–Tontiho konstrukce jsou studovány příklady globálně variačních zdrojových forem (diferenciálních rovnic) na otevřené Möbiově pásce. Podkladová varieta byla zvolena s ohledem na teorii variační posloupnosti, která zaručuje globální variačnost na topologických prostorech s triviální druhou de Rhamovou kohomologickou grupou.

Klíčová slova: Möbiova páska, Helmholtzovy podmínky, Vainberg–Tonti lagrangián,variační posloupnost, globální variačnost.