Numerical Approaches for Beams on Nonlinear Foundation - Part 2 (Applications)

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Abstract: This work presents numerical approaches and applications for the solutions of straight plane beams rested on elastic foundations. There are linear/nonlinear modified bilateral and unilateral Winkler's models (i.e. suitable approximations for dependencies of distributed reaction forces on deflection in the foundation). At first, for solutions of bilateral foundation, the Central Difference Method is applied in combination with the Newton's Method. At second, for solutions of unilateral foundation, the Finite Element Method is applied in combination with the Semi-Smooth Newton's method. The results acquired by linear/nonlinear approaches are evaluated and compared.

Keywords: unilateral and bilateral elastic foundation, nonlinear foundation, beam, Finite Element Method, Semi-Smooth Newton's method, Central Difference Method, programming.

1 Introduction

Beams on elastic foundations are often used in civil, mechanical and miming engineering, biomechanics etc.; see [4], [5], [7], [9]. This work is a continuation of our previous work where the theory is explained, see [9].

Hence, the symmetrical 2D Bernouli's beam of length 2L $=$ 2×12.045 m or 2L $=$ 2×6 m with cross-section $b \times h$ (b $=$ 0.2 m, h $=$ 0.4 m) is resting on an elastic foundation, see Fig. 1. The beam is loaded by force F $=$ 7×10^6 N. The modulus of elasticity of the beam is E $=$ 2×10^{11} Pa, the principal quadratic moment of the beam cross-section is $J_{ZT} = \frac{bh^3}{12}$ and the differential equation of this problem is $EJ_{ZT}\frac{d^4v}{dx^4} + q_R = 0$, where $q_R = q_R(x, v, \ldots)$ [Nm^{-1}] are the linear/nonlinear descriptions for reactions in the foundation and $v = v(x)$ [m] is deflection (i.e. vertical displacement).

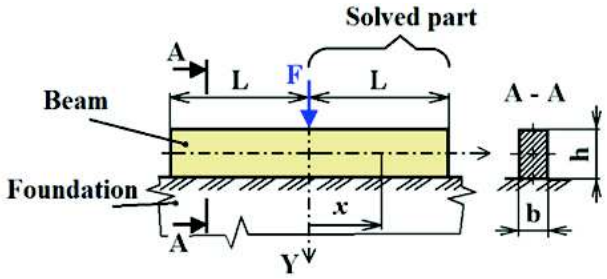

Figure 1: *Solved beam of length 2L resting on an elastic foundation.*

From the boundary conditions prescribed in points $x = 0$ m and $x = \mathrm{L}$ follow equations

$$\begin{aligned} &\frac{dv(x=0)}{dx} = 0 && && M_o(x=\mathrm{L}) = 0 \quad \Rightarrow \quad \frac{d^2v(x=\mathrm{L})}{dx^2} = 0, \\ &T(x=0) = \frac{\mathrm{F}}{2} \quad \Rightarrow \quad \frac{d^3v(x=0)}{dx^3} = \frac{\mathrm{F}}{2EJ_{ZT}} && && T(x=\mathrm{L}) = 0 \quad \Rightarrow \quad \frac{d^3v(x=\mathrm{L})}{dx^3} = 0. \end{aligned} \tag{1}$$

where $T(x)$ [N] is shearing force and M_0 [Nm] is bending moment.

In practical applications, the Central Difference Method (CDM) in combination with the Newton's Method and the Finite Element Method (FEM) in combination with the Semi-Smooth Newton's Method are applied.

2 Bilateral Elastic Foundation — Application of CDM in Combination with Newton's Method

In this case, the reaction in elastic bilateral foundation is described by three chosen functions q_{R_1}, $q_{R_{1,3}}$ and $q_{R_{1,3,5}}$ (i.e. linear approximation of experiments $\frac{d^4v}{dx^4} + \frac{k_1 v}{EJ_{ZT}} = 0$ with $k_1 = 2.3587 \times 10^7 \mathrm{Nm}^{-2}$, linear + cubic approximation of experiments $\frac{d^4v}{dx^4} + \frac{k_1 v + k_3 v^3}{EJ_{ZT}} = 0$ with $k_1 = 1.094 \times 10^7$ Nm^{-2}, $k_3 = 4.2869 \times 10^{12}$ Nm^{-4}, and (i.e. linear + cubic + quintic approximation of experiments, $\frac{d^4v}{dx^4} + \frac{k_1 v + k_3 v^3 + k_5 v^5}{EJ_{ZT}} = 0$ with $k_1 = 8.8597 \times 10^6$ Nm^{-2}, $k_3 = 6.4373 \times 10^{12}$ Nm^{-4}, $k_5 = -4.1846 \times 10^{17}$ Nm^{-6}); see [9].

According to CDM, the beam and its surroundings can be divided into $n+5$ nodes "i" with step $\Delta = \frac{\mathrm{L}}{n}$ for $x \in (0, \mathrm{L})$. For example, the differential equation $\frac{d^4v}{dx^4} + \frac{k_1 v + k_3 v^3 + k_5 v^5}{EJ_{ZT}} = 0$ can be approximated by CDM as

$$v_{i-2} - 4v_{i-1} + (6 + a_1)v_i - 4v_{i+1} + v_{i+2} + a_3 v_i^3 + a_5 v_i^5 = 0 \quad \text{for } i = 0, 1, 2, \ldots, n \tag{2}$$

where $a_1 = \frac{k_1 \Delta^4}{EJ_{ZT}}$, $a_3 = \frac{k_3 \Delta^4}{EJ_{ZT}}$, $a_5 = \frac{k_5 \Delta^4}{EJ_{ZT}}$, $c = 6 + a_1$ and $b = \frac{\mathrm{F}\Delta^3}{EJ_{ZT}}$, see [1], [2], [9].

From the boundary conditions (1) follow

$$\begin{aligned} &\frac{v_{i+1} - v_{i-1}}{2\Delta} = 0 \quad \Rightarrow \quad v_1 - v_{-1} = 0, \\ &\frac{v_{i+2} - 2v_i + 2v_{i-1} - v_{i-2}}{2\Delta^3} = \frac{\mathrm{F}}{2EJ_{ZT}} \quad \Rightarrow \quad v_2 - 2v_1 + 2v_{-1} - v_{-2} = b, \\ &\frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2} = 0 \quad \Rightarrow \quad v_{n+1} - 2v_n + v_{n-1} = 0, \\ &\frac{v_{i+2} - 2v_i + 2v_{i-1} - v_{i-2}}{2\Delta^3} = 0 \quad \Rightarrow \quad v_{n+2} - 2v_{n+1} + 2v_{n-1} - v_{n-2} = 0. \end{aligned} \tag{3}$$

The variables v_{-2}, v_{-1}, v_{n+1} and v_{n+2} (i.e. results in fictitious nodes -2, -1, $n+1$ and $n+2$, see [1], [2], [9]) can be expressed from boundary conditions (3). Hence, the set of nonlinear equations can be written in the matrix form as

$$\boldsymbol{M}\boldsymbol{v} + a_3\boldsymbol{v}^3 + a_5\boldsymbol{v}^5 - \boldsymbol{b} = \boldsymbol{o}, \tag{4}$$

$$\boldsymbol{b} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad \boldsymbol{v}^3 = \begin{pmatrix} v_0^3 \\ v_1^3 \\ \vdots \\ v_n^3 \end{pmatrix}, \quad \boldsymbol{v}^5 = \begin{pmatrix} v_0^5 \\ v_1^5 \\ \vdots \\ v_n^5 \end{pmatrix}.$$

$$\boldsymbol{M} = \begin{pmatrix} c & -8 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -4 & 7+a_1 & -4 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -4 & c & -4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -4 & c & -4 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 1 & -4 & c & -4 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -4 & c & -4 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & -4 & 5+a_1 & -2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 2 & -4 & 2+a_1 \end{pmatrix},$$

For more information see Part 1 of this article [9].

The system of coupled nonlinear equations (4) can be solved iteratively via Newton's (Newton–Raphson) Method as

$$\boldsymbol{v}^{(j+1)} = \boldsymbol{v}^{(j)} - \left(\boldsymbol{J}^{(j)}\right)^{-1}\left(\boldsymbol{M}\boldsymbol{v}^{(j)} + a_3\left(\boldsymbol{v}^{(j)}\right)^3 + a_5\left(\boldsymbol{v}^{(j)}\right)^5 + \boldsymbol{b}\right), \tag{5}$$

where vectors $\boldsymbol{v}^{(j)} = (v_0^{(j)}, v_1^{(j)}, \ldots, v_n^{(j)})^\top$ and $\boldsymbol{v}^{(j+1)} = (v_0^{(j+1)}, v_1^{(j+1)}, \ldots, v_n^{(j+1)})^\top$ are old and new iterations and $\boldsymbol{J}^{(j)} = \left(\frac{\partial f_i}{\partial v_k^{(j)}}\right)_{i,k=0,1,2,\ldots,n}$ is the matrix of derivatives (i.e. the Jacobian matrix) defined by

$$\boldsymbol{J}^{(j)} = \begin{pmatrix} J_{0,0} & -8 & 2 & 0 & 0 & \cdots & 0 \\ -4 & J_{1,1} & -4 & 1 & 0 & \cdots & 0 \\ 1 & -4 & J_{2,2} & -4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & -4 & J_{n-2,n-2} & -4 & 1 \\ 0 & \cdots & 0 & 1 & -4 & J_{n-1,n-1} & -2 \\ 0 & \cdots & 0 & 0 & 2 & -4 & J_{n,n} \end{pmatrix}, \tag{6}$$

where diagonal matrix elements $J_{0,0} = c + 3a_3(v_0^{(j)})^2 + 5a_5(v_0^{(j)})^4$, $J_{1,1} = 7 + a_1 + 3a_3(v_1^{(j)})^2 + 5a_5(v_2^{(j)})^4$, $J_{i,i} = c + 3a_3(v_i^{(j)})^2 + 5a_5(v_i^{(j)})^4$, for $i = 2, \ldots, n-2$, $J_{n-1,n-1} = 5 + a_1 + 3a_3(v_{n-1}^{(j)})^2 + 5a_5(v_{n-1}^{(j)})^4$ and $J_{n,n} = 2 + 3a_3(v_n^{(j)})^2 + 5a_5(v_n^{(j)})^4$.

Table 1: *Programming in MATLAB software*

```
% MATLAB (solution of a beam on an elastic nonlinear bilateral foundation)
% INPUTS; see previous text:
L=12.045; F=1e5; E=2e11; Jzt=(0.2*0.4^3)/12; v0=1e-4;
k1=8.8597e6; k3=6.4373e12;  k5=-4.1846e17;
n=500; accuracy=1e-5; delta=L/n; x=0:delta:L;
% PARAMETERS FOR SET OF EQUATIONS:
b1=(F*delta^3)/(E*Jzt); a1=(k1*delta^4)/(E*Jzt);
a3=(k3*delta^4)/(E*Jzt); c=6+a1; a5=(k5*delta^4)/(E*Jzt);
% BUILD OF BASIC MATRICES AND VECTORS:
Z=v0*ones(n+1,1); % initial deflection
Z3=Z.^3;  Z5=Z.^5;
M=[c  -8  2  0  0  0; -4  7+a1  -4  1  0  0];
M(n,n-2)=1; M(n,n-1)=-4; M(n,n)=5+a1; M(n,n+1)=-2;
M(n+1,n-2)=0; M(n+1,n-1)=2; M(n+1,n)=-4;  M(n+1,n+1)=2+a1;
for i=3:n-1
  M(i,i-2)=1; M(i,i-1)=-4; M(i,i)=c; M(i,i+1)=-4; M(i,i+2)=1;
end
b(1,1)=b1; b(n+1,1)=0;
% BUILD OF JACOBIAN:
for i=1:n+1, D(i,i)=3*a3*Z(i,1)^2+a5*Z(i,1)^4; end
J=M+D;                  % Jacobian matrix
% 1st ITERATION:
F=M*Z+a3*Z3+a5*Z5-b;
V=Z-J\F;                % deflection (first iteration)
error_of_deflection(1)=norm(V-Z);
% NEXT ITERATIONS:
G=1;                    % Number of iterations
while error_of_deflection(G) >= accuracy
    Z=V;  G=G+1;
    for i=1:n+1, Z3(i,1)=Z(i,1)^3;   Z5(i,1)=Z(i,1)^5; end
    F=M*Z+a3*Z3+a5*Z5-b;
    for i=1:n+1, D(i,i)=3*a3*Z(i,1)^2+a5*Z(i,1)^4; end
    J=M+D;              % Jacobian matrix
    V=Z-J\F;            % deflection (new iteration)
    error_of_deflection(G)=norm(V-Z);
end
% SLOPE dV:
dV(1,1)=0;  dV(n+1,1)=(2*V(n+1,1)-2*V(n,1))/(2*delta);
for i=2:n, dV(i,1)=(V(i+1,1)-V(i-1,1))/(2*delta); end
% BENDING MOMENT Mo:
Mo(1,1)=(-E*Jzt/delta^2)*(2*V(2,1)-2*V(1,1));  Mo(n+1,1)=0;
for i=2:n, Mo(i,1)=(-E*Jzt/delta^2)*(V(i+1,1)-2*V(i,1)+V(i-1,1)); end
% SHEARING FORCE T:
T(1,1)=(-E*Jzt/(2*delta^3))*b1;
T(2,1)=(-E*Jzt/(2*delta^3))*(V(4,1)-2*V(3,1)+2*V(1,1)-V(2,1));
T(n,1)=(-E*Jzt/(2*delta^3))*(-V(n,1)+2*V(n-1,1)-V(n-2,1));
T(n+1,1)=0;
for i=3:n-1
  T(i,1)=(-E*Jzt/(2*delta^3))*(V(i+2,1)-2*V(i+1,1)+2*V(i-1,1)-V(i-2,1));
end
```

The solution (iterative loops) was performed via MATLAB software; see the MATLAB script in Table 1. The value 10^{-5} m was chosen for the "accuracy". Just a few iterative loops were enough to generate accurate results.

Some basic results for the long beam (i.e. total beam length $2L = 2 \times 12.045$ m) are presented in Fig. 2 to 5 (i.e. dependencies for deflection, slope, bending moment and shearing force on coordinate x for different types of linear and nonlinear foundation). The distinctions between each type of foundation are evident. Deflection and slope (see Fig. 2 and 3) show the biggest differences between linear and nonlinear solutions. However, nonlinear approximations of foundations $q_{R_{1,3}}$ and $q_{R_{1,3,5}}$ give nearly the same results; see Fig. 2 to 5.

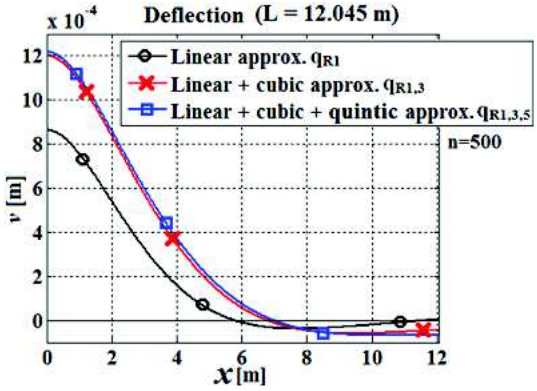

Figure 2: *Dependence for deflection on coordinate x of the beam for different types of approximations (solved example, bilateral foundation - results acquired by CDM).*

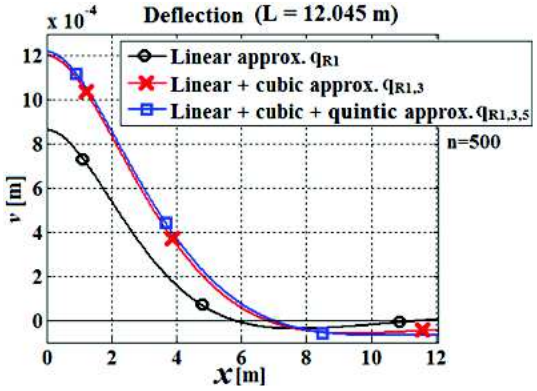

Figure 3: *Dependence for slope on coordinate x of the beam for different types of approximations (solved example, bilateral foundation - results acquired by CDM).*

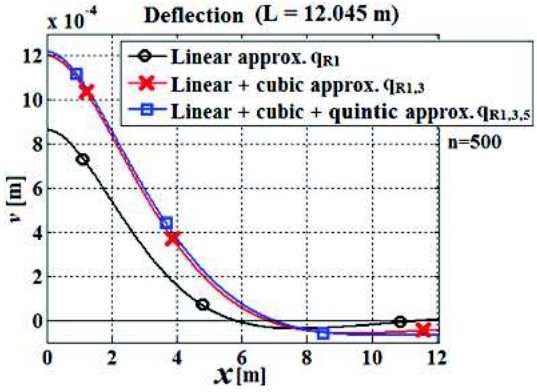

Figure 4: *Dependence for bending moment on coordinate x of the beam for different types of approximations (solved example, bilateral foundation - results acquired by CDM).*

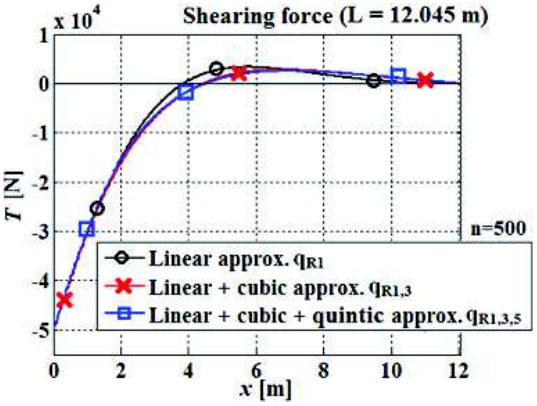

Figure 5: *Dependence for shearing force on coordinate x of the beam for different types of approximations (solved example, bilateral foundation - results acquired by CDM).*

Hence, from the results, the importance of the correct choice with regard to foundation behaviour is evident.

3 Unilateral Elastic Foundation – Application of FEM in Combination with Semi-Smooth Newton's Method

Let us suppose that the solved beam has symmetry. Therefore it is sufficient to solve the differential equation for a half of the beam, i.e. $x \in \langle 0; \mathrm{L}\rangle$.

Hence, the deflection of the beam is described by the equation

$$EJ_{ZT}\frac{d^4v}{dx^4} + kv^+ = 0 \qquad \text{on} \quad x \in (0, \mathrm{L})$$

with following boundary conditions prescribed in (1).

Lets divide the interval $(0, \mathrm{L})$ into n parts (elements) of the same length. This equidistant discretization with nodes $x_1 = 0, x_{i+1} = x_i + h$ has the constant step $h = \mathrm{L}/n$.

The discrete form of the weak formulation is following, see [9]

$$\begin{aligned} &\text{find } v_h \in V_h \text{ such that} \\ &EJ_{ZT}\int_0^{\mathrm{L}} \frac{d^2v_h}{dx^2}\frac{d^2\varphi_i}{dx^2}\,dx + k\int_0^{\mathrm{L}} v_h^+\varphi_i\,dx = \frac{\mathrm{F}}{2}\varphi_i(0) \text{ for all } i = 1, \ldots, 2n+2, \end{aligned} \tag{7}$$

where φ_i, $i = 1, \ldots 2n+2$ are piecewise-cubic smooth functions, the base function of space V_h. Because the solution v_h of (7) is element of the space $V_h = \left\{v_h \in C^1\big((0,\mathrm{L})\big) :\ v_h|_{\langle x_i, x_{i+1}\rangle} \in P_3,\ \frac{dv_h(0)}{dx} = 0\right\}$, we can write

$$v_h = \sum_{i=1}^{2n+2} u_i\varphi_i(x). \tag{8}$$

And we will denote the vector $\boldsymbol{u}$

$$\boldsymbol{u} = \left(v_h(x_1), \frac{dv_h(x_1)}{dx}, v_h(x_2), \frac{dv_h(x_2)}{dx}, \ldots, v_h(x_{n+1}), \frac{dv_h(x_{n+1})}{dx}\right)^\top.$$

The algebraic FEM representation of the first integral in (7) and the right side of (7) can be set by a standart way. The global stiffness matrix $\boldsymbol{K}$ and the global load vector $\boldsymbol{f}$ corresponding to (7) are shown. ($h = \mathrm{L}/n$ constant).

$$\boldsymbol{K} = \frac{1}{h^3}\begin{pmatrix} 12 & 0 & -12 & 6h & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & h^3 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ -12 & 0 & 24 & 0 & -12 & 6h & \ldots & 0 & 0 & 0 & 0 \\ 6h & 0 & 0 & 8h^2 & -6h & 2h^2 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6h & 24 & 0 & \ddots & -12 & 6h & 0 & 0 \\ 0 & 0 & 6h & 2h^2 & 0 & 8h^2 & \ddots & -6h & 2h^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6h & \ddots & 24 & 0 & -12 & 6h \\ 0 & 0 & 0 & 0 & 6h & 2h^2 & \ddots & 0 & 8h^2 & -6h & 2h^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & -12 & -6h & 12 & -6h \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 6h & 2h^2 & -6h & 4h^2 \end{pmatrix} \qquad \boldsymbol{f} = \begin{pmatrix} \frac{\mathrm{F}}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For more information see Part 1 of this article [5], [9].

The acquired FEM results (for L = 6 m, F = 10^5 N, $k = 2.3587 \times 10^7$ Nm^{-2}) were compared with the Central Difference Method with good agreement. There is a comparison of unilateral and bilateral linear approaches of elastic foundation. For example $v_{MAX,bilateral} = 9.0607 \times 10^{-4}$ m, $v_{MAX,unilateral} = 9.4242 \times 10^{-4}$ m, $M_{o\ MAX,bilateral} = 6.2070 \times 10^4$ Nm, $M_{o\ MAX,unilateral} = 6.6701 \times 10^4$ Nm. The differencies between unilateral and bilateral foundation are evident, see Fig. 6 to 10.

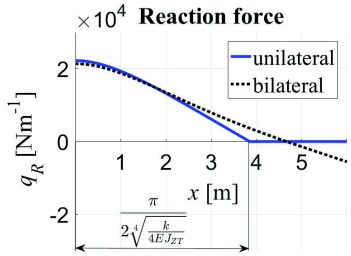

Figure 6: *Dependence for reaction force on coordinate x of the beam: Beam on unilateral (results acquired by FEM) and bilateral elastic foundation.*

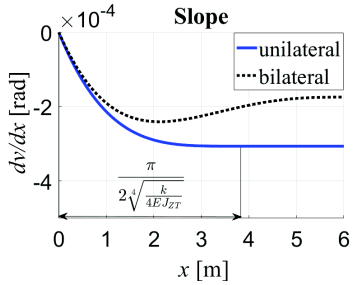

Figure 7: *Dependence for slopes on coordinate x of the beam: Beam on unilateral (results acquired by FEM) and bilateral elastic foundation.*

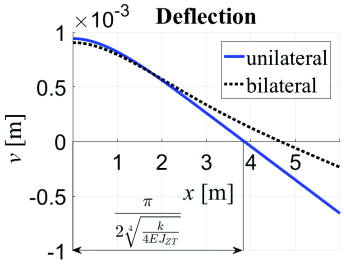

Figure 8: *Dependence for deflection on coordinate x of the beam: Beam on unilateral (results acquired by FEM) and bilateral elastic foundation.*

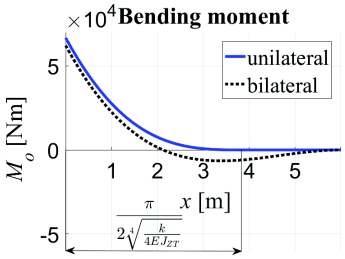

Figure 9: *Dependence for bending moment on coordinate x of the beam: Beam on unilateral (results acquired by FEM) and bilateral elastic foundation.*

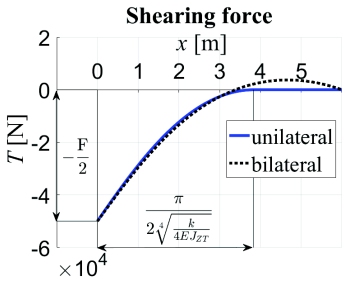

Figure 10: *Dependence for shearing force on coordinate x of the beam: Beam on unilateral (results acquired by FEM) and bilateral elastic foundation.*

Conclusion

This work is a continuation of our previous work (see Part 1 of this article [9]) presenting the theory and practice of beams on elastic linear/nonlinear foundations. Two applications (i.e. beam on bilateral foundation and beam on unilateral foundation) are presented. he CDM with Newton's method and FEM with semi-smooth Newton's method were used.

Acknowledgement

The authors gratefully acknowledge the funding from the Czech projects SP2016/145.

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Numerické řešení ohybu nosníku v nelineárním prostředí - část 1 (aplikace)

Abstrakt: V článku se zabýváme numerickým řešením úlohy, která popisuje ohyb rovinného nosníku uloženého v různých typech elastického prostředí, a to lineární/nelineární modifikované bilaterální a unilaterální Winklerova typu. Jsou popsány dva způsoby řešení této úlohy. První je založený na metodě centrálních diferencí a použití klasické Newtonovy metody a druhý je pomocí metody konečných prvků s využitím nehladké Newtonovy metody. V článku jsou uvedeny výsledky a srovnání těchto výpočetních postupů.

Klíčová slova: jednostranné a oboustranné elastické podloží, nelineární podloží, nosník, metoda konečných prvků, nehladká Newtonova metoda, metoda konečných diferencí, programování.