Systems of independent sets and systems of conditionally independent sets

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Abstract: System of conditionally independent sets is a generalization of the term system of independent sets. Definitions and examples of such systems and the overview of generalizations of results which led to the introduction of the term system of conditionally independent sets is described here. Some known results which could be generalized are mentioned.

Keywords: asymptotic density, system of independent sets, system of conditionally independent sets.

1 Introduction

1.1 Denotation

We use common denotation in this paper. Denotation $a \mid b$ means that number a divides b and $a \nmid b$ means, that number a does not divide b. We denote the greatest common divisor of numbers $a_1, \ldots, a_n$ by $gcd(a_1, \ldots, a_n)$, the least common multiple of numbers $a_1, \ldots, a_n$ by $\mathrm{lcm}(a_1, \ldots, a_n)$, the number of elements of the set $A \subseteq \mathbb{N}$ which are less or equal to n by $A(n)$, the lower asymptotic density of the set A by $\underline{d}(A)$, the upper asymptotic density of the set A by $\overline{d}(A)$, the asymptotic density of the set A by $d(A)$, and the difference of sets A and B by $A - B$. Moreover, for given increasing sequence of non-negative integers $\{\alpha(k)\}_{k=1}^{\infty}$ and $a \in \mathbb{N} - \{1\}$ we denote

$$A_a^{\alpha(k)} = \{a^{\alpha(k)}m : m, k \in \mathbb{N}, a \nmid m\}.$$

1.2 Asymptotic density

The upper asymptotic density of the set $A \subseteq \mathbb{N}$ is defined as the number $\bar{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$ and the lower asymptotic density of the set $A \subseteq \mathbb{N}$ is defined as the number $\underline{\mathrm{d}}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$ and the number $d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$ we call asymptotic density of the set A (if it exists).

Some of well known properties of asymptotic density, which we use below, are summed up in Lemma 1.1.

Lemma 1.1 *If $A \subseteq \mathbb{N}$, then holds:*

- *The upper and lower asymptotic density of set $A \subseteq \mathbb{N}$ exist, but the asymptotic density of A needs not.*
- $0 \leq \underline{d}(A) \leq \overline{d}(A) \leq 1$.
- $d(\mathbb{N}) = 1$.
- *If $d(A)$ exists, then $d(\mathbb{N} - A) = 1 - d(A)$,*
- *Let $c \in \mathbb{N}$, $B = \{ca : a \in A\}$. If $d(A)$ exists, then $d(B)$ exists too and*

$$d(B) = \frac{d(A)}{c}.$$

Proofs of these statements are trivial and follow immediately from the definition of the asymptotic density.

2 Systems of independent sets

The system of independent sets was defined as follows in [4].

Definition 2.1 *Let I be a set and $\{A_i\}_{i \in I}$ be a system of sets where $A_i \subseteq \mathbb{N}$ and $d(A_i)$ exists for every $i \in I$. System of sets $\{A_i\}_{i \in I}$ is a system of independent sets if for every finite set $F \subseteq I$ holds*

$$d\left(\bigcap_{i \in F} A_i\right) = \prod_{i \in F} d(A_i).$$

Asymptotic density of a set A_i can be interpreted as an estimation of the probability that we randomly choose an element from A_i in the set $\{1, 2, \ldots, n\}$ where n is "large" (see [7]). From this point of view, we can say that the events "to choose an element from A_i" are independent (within the meaning of the probability theory) in case of $\{A_i\}_{i \in I}$ is a system of independent sets.

Trivial example of system of independent sets is a system formed by sets $A_i \subseteq \mathbb{N}$, $i \in I$ where $d(A_i) = 0$. In this case, it is obvious that for every finite set $F \subseteq I$ holds

$$d\left(\bigcap_{i \in F} A_i\right) = \prod_{i \in F} d(A_i) = 0$$

We will use following criterion (see [2]) to create more examples of systems of independent sets.

Theorem 2.1 *System of sets $\left\{A_i\right\}_{i \in I}$ is a system of independent sets if and only if the system $\left\{\mathbb{N} - A_i\right\}_{i \in I}$ is a system of independent sets.*

Example 2.1 *System of sets* $\{A_i\}_{i\in I}$ *where sets* A_i *are defined below is a system of independent sets.*

1. $A_i = A_{p_i}^k = \{p_i m : m \in \mathbb{N}\}$

where $\{p_i\}_{i\in\mathbb{N}}$ *is a sequence of pairwise distinct prime numbers,* $I = \mathbb{N}$.

It is obvious (see Lemma1.1) that

$$d(A_i) = \frac{1}{p_i}$$

and $\bigcap\limits_{i\in F} A_i = \left\{ m \prod\limits_{i\in F} p_i : m \in \mathbb{N} \right\}$. *Hence*

$$d\left(\bigcap_{i\in F} A_i\right) = \frac{1}{\prod_{i\in F} p_i} = \prod_{i\in F} d(A_i).$$

2. $A_i = \mathbb{N} - A_{p_i}^k = \{m : m \in \mathbb{N}, \gcd(p_i, m) = 1\}$

where $\{p_i\}_{i\in\mathbb{N}}$ *is a sequence of pairwise distinct prime numbers,* $I = \mathbb{N}$.

This system of independent sets we obtain from previous item using Theorem 2.1.

3. $A_i = A_{p_i}^{2k} = \{p_i^{2k} m \mid k, m \in \mathbb{N}, \gcd(p_i, m) = 1\}$

where $\{p_i\}_{i\in\mathbb{N}}$ *is a sequence of pairwise distinct prime numbers,* $I = \mathbb{N}$. *It was proved in [5] that*

$$d(A_{p_i}^{2k}) = \frac{p_i}{p_i + 1}$$

and

$$d(\bigcap_{i\in F} A_{p_i}^{2k}) = \prod_{i\in F} \frac{p_i}{p_i + 1} = \prod_{i\in F} d(A_{p_i}^{2k}).$$

(This result can be used to determine the asymptotic density of the set B_2 *of natural numbers which are sum of two squares of integers (see [1]). But there are more general results describing* $B_2(n)$ *(see [8]).)*

4. $A_i = \mathbb{N} - A_{p_i}^{2(k-1)} = \{p_i^{2k-1} m \mid k, m \in \mathbb{N}, \gcd(p_i, m) = 1\}$

where $\{p_i\}_{i\in\mathbb{N}}$ *is a sequence of pairwise distinct prime numbers,* $I = \mathbb{N}$.

This system of independent sets we obtain from previous item using Theorem 2.1.

Asymptotic densities of sets $A_i^{\alpha(k)}$ and of their intersections were studied in [3]. It was found :

Theorem 2.2 *Let* $p \in \mathbb{N}$, $p > 1$, *and let* $\{\alpha(k)\}_{n=1}^{\infty}$ *be an increasing sequence of non-negative integers. If we denote* $A_p^{\alpha(k)} = \{p^{\alpha(k)} m \mid m, k \in \mathbb{N}, p \nmid m\}$, *then*

$$d(A_p^{\alpha(k)}) = \left(1 - \frac{1}{p}\right) \sum_{j=1}^{\infty} \frac{1}{p^{\alpha(j)}}.$$

and

Theorem 2.3 *Let $P = \{p_1, p_2, \ldots, p_r\}$ be a set of pairwise co-prime natural numbers where $1 \notin P$ and $\{\alpha_1(k)\}_{k=1}^{\infty}, \{\alpha_2(k)\}_{k=1}^{\infty}, \ldots, \{\alpha_r(k)\}_{k=1}^{\infty}$ are increasing sequences of non-negative integers. Then*

$$d\left(\bigcap_{i=1}^{r} A_{p_i}^{\alpha_i(k)}\right) = \prod_{i=1}^{r} d(A_{p_i}^{\alpha_i(k)}).$$

Theorem 2.2 and Theorem 2.3 imply:

Theorem 2.4 *Let $\{p_i : i \in I\}$ be a nonempty set of pairwise co-prime natural numbers where for every $i \in I$ holds $p_i > 1$ and $\{\alpha_i(k)\}_{k=1}^{\infty}$ is an increasing sequence of non-negative integers. Then $\left\{A_{p_i}^{\alpha_i(k)}\right\}_{i\in I}$ is a system of independent sets.*

Asymptotic density of the intersection of sets $A_a^{\alpha(n)} = \{a^{\alpha(n)}m : m, n \in \mathbb{N}, a \nmid m\}$ and $A_b^{\beta(n)} = \{b^{\beta(n)}m : m, n \in \mathbb{N}, b \nmid m\}$ where $\{\alpha(n)\}_{n=1}^{\infty}$ and $\{\beta(n)\}_{n=1}^{\infty}$ are given increasing sequences of natural numbers, $a, b \in \mathbb{N} - \{1\}$ and $\gcd(a, b) = d \in \mathbb{N}$ was studied in [4]. It was found that

$$d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{d^{\min\{\alpha_i,\beta_j\}}}{a^{\alpha_i} b^{\beta_j}} \left(1 - d(C_{ij}) - d(D_{ij}) + d\left(C_{ij} \cap D_{ij}\right)\right) \tag{1}$$

where

$$\begin{aligned} d\left(C_{ij}\right) &= \frac{\gcd(a\cdot\gcd(a^{\alpha_i}, b^{\beta_j}), b^{\beta_j})}{a\cdot\gcd(a^{\alpha_i}, b^{\beta_j})} = \frac{\gcd(a\cdot d^{\min\{\alpha_i,\beta_j\}}, b^{\beta_j})}{a\cdot d^{\min\{\alpha_i,\beta_j\}}}, \\ d\left(D_{ij}\right) &= \frac{\gcd(b\cdot\gcd(a^{\alpha_i}, b^{\beta_j}), a^{\alpha_i})}{b\cdot\gcd(a^{\alpha_i}, b^{\beta_j})} = \frac{\gcd(b\cdot d^{\min\{\alpha_i,\beta_j\}}, a^{\alpha_i})}{b\cdot d^{\min\{\alpha_i,\beta_j\}}}, \\ d\left(C_{ij} \cap D_{ij}\right) &= d\left(C_{ij}\right) d\left(D_{ij}\right) \gcd\left(\tfrac{1}{d(C_{ij})}, \tfrac{1}{d(D_{ij})}\right). \end{aligned}$$

This result implies that in case of a and b are not co-primes sets $A_a^{\alpha(n)}$ and $A_b^{\beta(n)}$ need not create system of independent sets.

The aim is to find result analogical to (1) for finite intersection of sets $A_{p_i}^{\alpha_i(k)}$, $i \in I$. It means, we are looking for the generalization of Theorem2.3 where numbers p_i need not be co-primes. The problem is that it is not possible to obtain this result analogically as (1) was obtained in [4].

Nevertheless we claim that the asymptotic density of the set $X = \bigcap_{k=1}^{p} A_{a_k}^{\alpha_k(n)}$ exists for arbitrary $p \in \mathbb{N}$ and it holds:

Conjecture 2.1 *Let $a_1, a_2, \ldots, a_p \in \mathbb{N} - \{1\}$. And let us define for every $k \in \{1, 2, \ldots, p\}$ sets $A_k^{\alpha_k} = \left\{a_k^{\alpha_k(i_k)}m : i_k \in \mathbb{N}, m \in \mathbb{N} - \mathbb{N}_{a_k}\right\}$, where $\mathbb{N}_{a_k} = \{a_k m : m \in \mathbb{N}\}$ and $\{\alpha_1(i_1)\}_{i_1=1}^{\infty}$, $\{\alpha_2(i_2)\}_{i_2=1}^{\infty}$, $\ldots$, $\{\alpha_p(i_p)\}_{i_p=1}^{\infty}$ be increasing sequences of natural numbers.*

If $X = \bigcap_{k=1}^{p} A_k^{\alpha_k}$, then $d(X)$ exists and

$$d(X) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_p=1}^{\infty} d\left(X_{i_1,\ldots,i_p}\right),$$

where

$$d\left(X_{i_1,\ldots,i_p}\right) = \frac{1}{\operatorname{lcm}\left(a_1^{\alpha_1(i_1)},\ldots,a_p^{\alpha_p(i_p)}\right)}\left(1-\sum_{s=1}^{p}\left((-1)^{(s-1)}\sum_{\{j_1,\ldots,j_s\}\subseteq\{1,\ldots,p\}} d_{i_1,\ldots,i_p}^{j_1,\ldots,j_s}\right)\right),$$

$$d_{i_1,\ldots,i_p}^{j_1,\ldots,j_s} = \frac{1}{\operatorname{lcm}(\delta_{j_1,i_1,\ldots,i_p},\delta_{j_2,i_1,\ldots,i_p},\ldots,\delta_{j_s,i_1,\ldots,i_p})}$$

and

$$\delta_{j_m,i_1,\ldots,i_p} = \frac{\operatorname{lcm}\left(a_{j_m}^{\alpha_{j_m}(i_{j_m})+1},\operatorname{lcm}\left(a_1^{\alpha_1(i_1)},\ldots,a_p^{\alpha_p(i_p)}\right)\right)}{\operatorname{lcm}\left(a_1^{\alpha_1(i_1)},\ldots,a_p^{\alpha_p(i_p)}\right)}.$$

3 Systems of conditionally independent sets

The system of independent sets was defined as follows in [6].

Definition 3.1 *Let I be a set, $B\subseteq\mathbb{N}$, $d(B)\neq 0$ exists and $\{A_i\}_{i\in I}$ be a system of sets where $A_i\subseteq\mathbb{N}$ and $d(A_i\cap B)$ exists for every $i\in I$. The system of sets $\{A_i\}_{i\in I}$ is a system of conditionally independent sets under condition B if and only if for every finite nonempty set $F\subseteq I$ holds*

$$\frac{d\left(\bigcap_{i\in F}(A_i\cap B)\right)}{d(B)} = \prod_{i\in F}\frac{d(A_i\cap B)}{d(B)}.$$

We also say, that sets A_i, where $i\in I$ are conditionally independent.

We can see that systems of independent sets are a special case of systems of conditionally independent systems. This is the trivial case where the condition is the set $B=\mathbb{N}$.

Several examples and ways of construction of systems of conditionally independent sets were found in [6]. Following theorems were proved:

Theorem 3.1 *Let $a,b_1,b_2\in\mathbb{N}$, $\gcd(a,b_1)=\gcd(a,b_2)=1$ and $p_1<p_2<\cdots<p_i<\ldots$ be sequence of prime numbers fulfilling $p_i\equiv b_1(\mod a)$ for every $i\in\mathbb{N}$. Let us denote*

$$A_i = \{p_i m_2 \mid m_2\in\mathbb{N}, m_2\equiv b_2(\mod a)\}$$

for every $i\in\mathbb{N}$. Then $\{A_i\}_{i\in\mathbb{N}}$ is a system of conditionally independent sets under condition $B=\{m_1m_2 \mid m_1,m_2\in\mathbb{N}; m_1\equiv b_1(\mod a); m_2\equiv b_2(\mod a)\}$.

This theorem shows particular system of conditionally independent sets.

Theorem 3.2 *Let I contain at least two elements, $\{A_i\}_{i\in I}$ is system of independent sets, $K\subset I$ is finite nonempty set and $B=\cap_{i\in K}A_i$ where $d(B)\neq 0$. Then $\{A_i\}_{i\in I-K}$ is a system of independent sets and also it is a system of conditionally independent sets under condition B.*

If we have given system of independent sets, then this theorem allows us to create system of conditionally independent sets.

Another way to create a system of conditionally independent sets from the system of independent sets $\{A_i\}_{i\in I}$ is that we put all the elements $a\in\bigcup_{i\in I}A_i$ to the transformation $a^*=ca+d$, where c and d are given natural numbers.

Theorem 3.3 *Let* $\{A_i\}_{i\in I}$ *is a system of independent sets,* $c \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$, $B = \{ca + d \mid a \in \mathbb{N}\}$ *and for every* $i \in I$ *holds* $B_i = \{ca + d \mid a \in A_i\}$. *Then* $\{B_i\}_{i\in I}$ *is a system of conditionally independent sets under condition* B.

But there was not introduced any criterion for systems of conditionally independent sets in [6]. Nevertheless, we are convinced now that Theorem2.1 can be generalized following way:

Conjecture 3.1 *System of sets* $\left\{A_i\right\}_{i\in I}$ *is a system of conditionally independent sets under condition* B *if and only if the system* $\left\{\mathbb{N} - A_i\right\}_{i\in I}$ *is a system of conditionally independent sets under condition* B.

Conclusion

We have described basic known properties of systems of independent sets and systems of conditionally independent sets. We have also introduced two conjectures. The first one generalizes result describing the asymptotic density of intersection of two sets of type $A_a^{\alpha(n)}$ and describes asymptotic density of intersection of arbitrary finite number of sets of type $A_a^{\alpha(n)}$. The second one generalizes a criterion of systems of independent sets and it is a criterion of systems of conditionally independent sets.

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