Systems of independent sets - results and conjectures

JAHODOVÁ Monika

VŠB-TU Ostrava, 17. listopadu 15/2172, 708 33 Ostrava-Poruba, CZ
E-mail: monika.jahodova@vsb.cz

Abstract: The system of independent sets is such a system of sets of natural numbers where the asymptotic density of the intersection of its arbitrary finite subsystem is equal to the product of asymptotic densities of sets which belong to this subsystem. Basic properties of systems of independent sets and some results and conjectures concerning them are described here.

Keywords: asymptotic density, system of independent sets.

1 Introduction

Terms probability, independence of random events and independence of random variables belong among basic terms of the probability theory, which are plentifully used in practice.

Lots of mathematical theorems and ideas are based on the assumption of independence of random events and random variables. For example, such assumptions are important in definitions of chi-squared distribution, binomial distribution etc. The well known central limit theorem, which describes distribution of the sum of independent random variables, is used in practice for approximation of various calculations.

From the other side, one of the most frequently used version of chi-square test verifies the hypothesis about independence of random variables. Concepts asymptotic density and system of independent sets are analogies of terms probability and independent events in number theory (see [8], Chapter 4).

The classical probability that a randomly chosen number from the set $\{n \in \mathbb{N} : \ n \leq n_0\}$ belongs to a set $A \subseteq \mathbb{N}$ can be often approximated for large number

n_0 by the asymptotic density of the set A.

Asymptotic density of the set $A \subseteq \mathbb{N}$ is defined as follows.

Definition 1.1. *Let $A \subseteq \mathbb{N}$. Number of elements of the set $A \subseteq \mathbb{N}$ which are less or equal to n is denoted by $A(n)$. Number $\bar{d}(A)$, where*

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n},$$

is called upper asymptotic density of set A and number $\underline{d}(A)$ where

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$$

is called lower asymptotic density of set A.

If $\bar{d}(A) = \underline{d}(A)$, then value $d(A) = \bar{d}(A) = \underline{d}(A) = \lim_{n\to\infty} \frac{A(n)}{n}$ is called asymptotic density of set A.

We immediately obtain from the definition the following basic properties of asymptotic density, which are further used:

- The upper and lower asymptotic density of set $A \subseteq \mathbb{N}$ certainly exists, but the asymptotic density of A may not.
- $0 \leq \underline{d}(A) \leq \overline{d}(A) \leq 1$
- $d(\mathbb{N}) = 1$

2 Systems of independent sets – results

The term "system of sets with multiplicative asymptotic density" was introduced in paper [3] in 2008. A more general term "system of independent sets" was later introduced in paper [5]. Its basic properties and particular examples were studied in [4], [6] and [5]. System of independent sets is an analogy of systems of independent events which are known from probability theory – the probability of their intersection is equal to the product of their probabilities. The system of independent sets is such a system of sets of natural numbers where the asymptotic density of the intersection of its arbitrary finite subsystem is equal to the product of asymptotic densities of sets which belong to this subsystem.

Definition 2.1. *Let I be a set and $\{A_i\}_{i\in I}$ be a system of sets where $A_i \subseteq \mathbb{N}$ and $d(A_i)$ exists for every $i \in I$. System of sets $\{A_i\}_{i\in I}$ is a system of independent sets if for every finite set $F \subseteq I$ holds*

$$d\left(\bigcap_{i\in F} A_i\right) = \prod_{i\in F} d(A_i).$$

A system of sets with the multiplicative asymptotic density was in [3] defined for $I = \mathbb{N}$. According to the above mentioned definition, systems of independent sets, contrary to systems of sets with the multiplicative asymptotic density, may also be definite, or uncountable. However, this generalization keeps in effect the Theorem 2.1 from article [3] also for systems of independent sets. We can reformulate it this way:

Theorem 2.1. *System of sets $\{A_i\}_{i\in I}$ is a system of independent sets if and only if the system $\{\mathbb{N} - A_i\}_{i\in I}$ is a system of independent sets.*

This criterion was generalized in [5] as follows:

Theorem 2.2. *The system $\{A_i\}_{i\in I}$ is a system of independent sets if and only if the system $\{B_i\}_{i\in I}$ where for every $i \in I$ holds $B_i = A_i$ or $B_i = \mathbb{N} - A_i$ is a system of independent sets.*

In paper [7], we have among others studied properties of a system of certain sets where set of natural numbers which are expressible in the form of sum of two squares of integers is included (you can see [2], [9], [10] for more information about this interesting set).

Let us denote $p_1 < p_2 < \cdots < p_i < \ldots$ the sequence of prime numbers in form $p_i = 4k+3$, where $k \in \mathbb{N}\cup\{0\}$ and $A_{p_i}^{2j} = \{m.p_i^{2j} \mid j \in \mathbb{N}\cup\{0\}, m \in \mathbb{N}, \gcd(m, p_i) = 1\}$. It means, that $A_{p_i}^{2j}$ is a set of those natural numbers, which have not the prime number p_i with odd exponent in their canonical decompositions. It is well known (see [1]), that set D cotaining all natural numbers, which are expressible in the form of sum of two squares of integers is equal to the intersection of sets $A_{p_i}^{2j}$, $i = 1, 2, \ldots$, ie. $D = \cap_{i=1}^{\infty} A_{p_i}^{2j}$.

We have proved following two statements in [7]:

Lemma 2.1. *Let us denote $p_1 < p_2 < \cdots < p_i < \ldots$ the sequence of prime numbers in form $p_i = 4k + 3$ where $k \in \mathbb{N} \cup \{0\}$ and $A_{p_i}^{2j} = \{m.p_i^{2j} \mid j \in \mathbb{N} \cup \{0\}, m \in \mathbb{N}, \gcd(m, p_i) = 1\}$. Then for asymptotic density of the set $A_{p_i}^{2j}$ holds*

$$d(A_{p_i}^{2j}) = \frac{p_i}{p_i + 1}.$$

and

Lemma 2.2. *Let $A_{p_i}^{2j}$ be sets defined in the same way as in previous lemma. Then for every $n \in \mathbb{N}$ holds that*

$$d(\bigcap_{i=1}^{n} A_{p_i}^{2j}) = \prod_{i=1}^{n} d(A_{p_i}^{2j}) = \prod_{i=1}^{n} \frac{p_i}{p_i + 1}.$$

We can see that the asymptotic density of the intersection of sets $A_{p_i}^{2j}, \ldots, A_{p_n}^{2j}$ is equal to the product of their asymptotic densities. If we would make minor alterations in the proof of Lemma 2.2, then we would find that for arbitrary finite subsystem of system $\left\{A_{p_i}^{2j}\right\}_{i=1}^{\infty}$ holds that the asymptotic density of intersection of its sets is equal to the product of their asymptotic densities.

This led us to study sets

$$A_a^{\alpha_n} = \{a^{\alpha_n} m : \ m, n \in \mathbb{N}; a \nmid m\}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is given increasing sequence of non-negative integers and $a \in \mathbb{N} - \{1\}$.

It has been shown in paper [4] that asymptotic densities of sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ exist and following theorem has been proven in [5].

Theorem 2.3. *Let $a, b \in \mathbb{N}$, $a, b \neq 1$, $\gcd(a, b) = d$ and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be increasing sequences of natural numbers. If $A_a^{\alpha_n} = \{a^{\alpha_n} m : \ m, n \in \mathbb{N}; a \nmid m\}$ and $A_b^{\beta_n} = \{b^{\beta_n} m : \ m, n \in \mathbb{N}; b \nmid m\}$, then*

$$d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{d^{\min\{\alpha_i, \beta_j\}}}{a^{\alpha_i} b^{\beta_j}} \left(1 - d(C_{ij}) - d(D_{ij}) + d\left(C_{ij} \cap D_{ij}\right)\right),$$

where

$$\begin{aligned} d\left(C_{ij}\right) &= \frac{\gcd(a \cdot \gcd(a^{\alpha_i}, b^{\beta_j}), b^{\beta_j})}{a \cdot \gcd(a^{\alpha_i}, b^{\beta_j})} = \frac{\gcd(a \cdot d^{\min\{\alpha_i, \beta_j\}}, b^{\beta_j})}{a \cdot d^{\min\{\alpha_i, \beta_j\}}}, \\ d\left(D_{ij}\right) &= \frac{\gcd(b \cdot \gcd(a^{\alpha_i}, b^{\beta_j}), a^{\alpha_i})}{b \cdot \gcd(a^{\alpha_i}, b^{\beta_j})} = \frac{\gcd(b \cdot d^{\min\{\alpha_i, \beta_j\}}, a^{\alpha_i})}{b \cdot d^{\min\{\alpha_i, \beta_j\}}}, \qquad (1) \\ d\left(C_{ij} \cap D_{ij}\right) &= d\left(C_{ij}\right) d\left(D_{ij}\right) \gcd\left(\tfrac{1}{d(C_{ij})}, \tfrac{1}{d(D_{ij})}\right). \end{aligned}$$

Example 2.1. *Let us consider the sets $A = \{6m \mid m \in \mathbb{N}\}$ and $B = \{15m \mid m \in \mathbb{N}\}$. It is clear, that their intersection is a set $A \cap B = \{30m \mid m \in \mathbb{N}\}$ and $d(A \cap B) = \frac{1}{30}$. We can also determine this value using the Theorem 2.3, but the procedure will be quite fiddly. The price, however, is in its universality. We can use it to determine (and we can be sure of its existence) the asymptotic density of the intersections of the more general sets than sets of multiples of given numbers.*

Let $A = \{6m \mid m \in \mathbb{N}\}$ and $B = \{15m \mid m \in \mathbb{N}\}$. Hence $A = A_a^{\alpha_n}$ a $B = A_b^{\beta_n}$, where $a = 6$, $b = 15$ and for every $n \in \mathbb{N}$ holds $\alpha_n = n$ and $\beta_n = n$.

According to the Theorem 2.3 holds

$$d(A \cap B) = \sum_{i=1}^{\infty} \sum_{i=j}^{\infty} d(X_{ij}), \qquad (2)$$

where

$$d(X_{ij}) = \frac{\gcd(a^{\alpha_i}, b^{\beta_j})}{a^{\alpha_i} b^{\beta_j}} \left(1 - d(C_{ij}) - d(D_{ij}) + d\left(C_{ij} \cap D_{ij}\right)\right), \tag{3}$$

By inserting specific values, we receive the following

$$\frac{\gcd(a^{\alpha_i}, b^{\beta_j})}{a^{\alpha_i} b^{\beta_j}} = \frac{\gcd(6^i, 15^j)}{6^i 15^j} = \frac{3^{\min\{i,j\}}}{6^i 15^j}$$

$$d\left(C_{ij}\right) = \frac{\gcd(6 \cdot \gcd(6^i,\, 15^j), 15^j)}{6 \cdot \gcd(6^i, 15^j)} = \frac{\gcd(2 \cdot 3^{\min\{i,j\}+1},\, 5^j 3^j)}{2 \cdot 3^{\min\{i,j\}+1}} \tag{4}$$

$$d\left(D_{ij}\right) = \frac{\gcd(15 \cdot \gcd(6^i, 15^j),\, 6^i)}{15 \cdot \gcd(6^i, 15^j)} = \frac{\gcd(5 \cdot 3^{\min\{i,j\}+1},\, 2^i 3^i)}{5 \cdot 3^{\min\{i,j\}+1}}$$

From (4) and (1) then follows

$$\frac{\gcd(a^{\alpha_i}, b^{\beta_j})}{a^{\alpha_i} b^{\beta_j}} = \begin{cases} (for\ i > j) = & \frac{3^j}{6^i 15^j} = \left(\frac{1}{6}\right)^i \left(\frac{1}{5}\right)^j \\ (for\ i = j) = & \frac{3^i}{6^i 15^i} = \left(\frac{1}{30}\right)^i \\ (for\ i < j) = & \frac{3^i}{6^i 15^j} = \left(\frac{1}{2}\right)^i \left(\frac{1}{15}\right)^j \end{cases}$$

$$d\left(C_{ij}\right) = \begin{cases} (for\ i > j) = & \frac{\gcd(2 \cdot 3^{j+1},\, 5^j 3^j)}{2 \cdot 3^{j+1}} = \frac{1}{6} \\ (for\ i = j) = & \frac{\gcd(2 \cdot 3^{i+1},\, 5^i 3^i)}{2 \cdot 3^{i+1}} = \frac{1}{6} \\ (for\ i < j) = & \frac{\gcd(2 \cdot 3^{i+1},\, 5^j 3^j)}{2 \cdot 3^{i+1}} = \frac{1}{2} \end{cases}$$

$$d\left(D_{ij}\right) = \begin{cases} (for\ i > j) = & \frac{\gcd(5 \cdot 3^{j+1},\, 2^i 3^i)}{5 \cdot 3^{j+1}} = \frac{1}{5} \\ (for\ i = j) = & \frac{\gcd(5 \cdot 3^{i+1},\, 2^i 3^i)}{5 \cdot 3^{i+1}} = \frac{1}{15} \\ (for\ i < j) = & \frac{\gcd(5 \cdot 3^{i+1},\, 2^i 3^i)}{5 \cdot 3^{i+1}} = \frac{1}{15} \end{cases}$$

$$d\left(C_{ij} \cap D_{ij}\right) = \begin{cases} (for\ , i > j) = & \frac{1}{6}\frac{1}{5} \gcd(6, 5) = \frac{1}{30} \\ (for\ i = j) = & \frac{1}{6}\frac{1}{15} \gcd(6, 15) = \frac{1}{30} \\ (for\ i < j) = & \frac{1}{2}\frac{1}{15} \gcd(2, 15) = \frac{1}{30} \end{cases}$$

Substituting into the equation (3) we receive

$$\begin{aligned} & & (for\ i > j) &= \left(\frac{1}{6}\right)^i \left(\frac{1}{5}\right)^j \frac{10}{15} \\ d(X_{ij}) &= & (for\ i = j) &= \left(\frac{1}{30}\right)^i \frac{12}{15} \\ & & (for\ i < j) &= \left(\frac{1}{2}\right)^i \left(\frac{1}{15}\right)^j \frac{7}{15} \end{aligned} \tag{5}$$

For $k \in \mathbb{N}$ let us denote $S(k) = \sum_{j=1}^{\infty} d(X_{kj})$. Hence

$$\begin{aligned} S(1) &= 0 + d(X_{11}) + \sum_{j=2}^{\infty} d(X_{1j}) = \\ &= 0 + \left(\frac{1}{30}\right)^1 \frac{12}{15} + \left(\frac{1}{2}\right)^1 \frac{7}{15} \sum_{j=2}^{\infty} \left(\frac{1}{15}\right)^j = \\ &= 0 + \left(\frac{1}{30}\right)^1 \frac{12}{15} + \left(\frac{1}{2}\right)^1 \frac{7}{15} \left(\frac{1}{15}\right)^2 \frac{15}{14} = \\ &= \frac{1}{36} \end{aligned}$$

$$\begin{aligned} S(2) &= d(X_{21}) + d(X_{22}) + \sum_{j=3}^{\infty} d(X_{2j}) = \\ &= \left(\frac{1}{6}\right)^2 \left(\frac{1}{5}\right)^1 \frac{10}{15} + \left(\frac{1}{30}\right)^2 \frac{12}{15} + \left(\frac{1}{2}\right)^2 \frac{7}{15} \sum_{j=3}^{\infty} \left(\frac{1}{15}\right)^j = \\ &= \left(\frac{1}{6}\right)^2 \left(\frac{1}{5}\right)^1 \frac{10}{15} + \left(\frac{1}{30}\right)^2 \frac{12}{15} + \left(\frac{1}{2}\right)^2 \frac{7}{15} \left(\frac{1}{15}\right)^3 \frac{15}{14} \end{aligned}$$

$$
\begin{aligned}
S(3) &= d(X_{31}) + d(X_{32}) + d(X_{33}) + \sum_{j=4}^{\infty} d(X_{3j}) = \\
&= \left(\frac{1}{6}\right)^3 \frac{10}{15}\left[\left(\frac{1}{5}\right)^1 + \left(\frac{1}{5}\right)^2\right] + \left(\frac{1}{30}\right)^3 \frac{12}{15} + \left(\frac{1}{2}\right)^3 \frac{7}{15} \sum_{j=4}^{\infty} \left(\frac{1}{15}\right)^j = \\
&= \left(\frac{1}{6}\right)^3 \frac{10}{15}\left[\left(\frac{1}{5}\right)^1 + \left(\frac{1}{5}\right)^2\right] + \left(\frac{1}{30}\right)^2 \frac{12}{15} + \left(\frac{1}{2}\right)^2 \frac{7}{15} \left(\frac{1}{15}\right)^4 \frac{15}{14} \\
&\vdots
\end{aligned}
$$

In general, for $k \geq 2$ we can write

$$
\begin{aligned}
S(k) &= \sum_{j=1}^{k-1} d(X_{kj}) + d(X_{kk}) + \sum_{j=k+1}^{\infty} d(X_{kj}) = \\
&= \sum_{j=1}^{k-1} \left(\frac{1}{6}\right)^k \left(\frac{1}{5}\right)^j \frac{10}{15} + \left(\frac{1}{30}\right)^k \frac{12}{15} + \sum_{j=k+1}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{15}\right)^j \frac{7}{15} = \\
&= \frac{10}{15}\left(\frac{1}{6}\right)^k \sum_{j=1}^{k-1} \left(\frac{1}{5}\right)^j + \left(\frac{1}{30}\right)^k \frac{12}{15} + \frac{7}{15}\left(\frac{1}{2}\right)^k \left(\frac{1}{15}\right)^{k+1} \frac{15}{14} = \\
&= \frac{10}{15}\left(\frac{1}{6}\right)^k \left[\frac{1}{5}\frac{1-\left(\frac{1}{5}\right)^{k-1}}{1-\frac{1}{5}}\right] + \left(\frac{1}{30}\right)^k \frac{24}{30} + \frac{1}{30}\left(\frac{1}{2}\right)^k \left(\frac{1}{15}\right)^k = \\
&= \frac{1}{6}\left(\frac{1}{6}\right)^k \left[1 - 5\left(\frac{1}{5}\right)^k\right] + \left(\frac{1}{30}\right)^k \frac{25}{30} = \\
&= \frac{1}{6}\left(\frac{1}{6}\right)^k - \frac{5}{6}\left(\frac{1}{30}\right)^k + \left(\frac{1}{30}\right)^k \frac{5}{6} = \\
&= \frac{1}{6}\left(\frac{1}{6}\right)^k
\end{aligned}
$$

Hence

$$
\begin{aligned}
d(A \cap B) &= \sum_{i=1}^{\infty}\sum_{i=j}^{\infty} d(X_{ij}) = \sum_{i=1}^{\infty} S(i) = S(1) + \sum_{i=2}^{\infty} \frac{1}{6}\left(\frac{1}{6}\right)^i = \\
&= \frac{1}{36} + \frac{1}{6}\frac{1}{6^2}\frac{1}{\left(1-\frac{1}{6}\right)} = \frac{1}{36}\left(1 + \frac{1}{5}\right) = \mathbf{\frac{1}{30}}.
\end{aligned}
$$

3 Systems of independent sets – conjectures

We have defined sets

$$A_a^{\alpha_n} = \{a^{\alpha_n} m : \ m, n \in \mathbb{N}; a \nmid m\}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is given increasing sequence of non-negative integers and $a \in \mathbb{N} - \{1\}$.

Sequence $\{\alpha_i\}_{i=1}^{\infty}$ is an increasing sequence of non-negative integers, therefore we can choose $\alpha_i = i - 1$. Then $A_a^{\alpha_i} = \mathbb{N}$ and for an arbitrary set $A_b^{\beta_j}$ then holds

$$d\left(A_a^{\alpha_i} \cap A_b^{\beta_j}\right) = d\left(\mathbb{N} \cap A_b^{\beta_j}\right) = d\left(A_b^{\beta_j}\right) = d\left(\mathbb{N}\right) d\left(A_b^{\beta_j}\right) = d\left(A_a^{\alpha_i}\right) d\left(A_b^{\beta_j}\right).$$

We can see that sets $A_a^{\alpha_i}$ and $A_b^{\beta_j}$ form a system of independent sets. But this case is trivial and not very interesting. If sequences $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ are increasing sequences of natural numbers, the above described construction of a system of independent sets fails because in this case $A_a^{\alpha_i} \subseteq \{am : m \in \mathbb{N}\} \neq \mathbb{N}$ and $A_b^{\beta_j} \subseteq \{bm : m \in \mathbb{N}\} \neq \mathbb{N}$.

The question is whether there exist numbers $a, b \in \mathbb{N} - \{1\}$ and sequences of natural numbers $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ so that sets $A_a^{\alpha_i}$ and $A_b^{\beta_j}$ form a system of independent sets.

The asymptotic density of the intersection of set $A_a^{\alpha_n} = \{a^{\alpha_n} m : \ m, n \in \mathbb{N}; a \nmid m\}$ and of the set $A_b^{\beta_n} = \{b^{\beta_n} m : \ m, n \in \mathbb{N}; b \nmid m\}$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are given increasing sequences of natural numbers and $\gcd(a, b) = d \in \mathbb{N}$ was investigated in paper [5].

It was stated that in case of $\gcd(a, b) = 1$ sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ form a system of independent sets, but in general case $\gcd(a, b) = d \in \mathbb{N}$, sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ may not form a system of independent sets.

We introduce following conjectures now.

Conjecture 3.1. *Let $a, b \in \mathbb{N}$; $a, b \neq 1$; $1 < \gcd(a, b) = d \neq \min\{a, b\}$ and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be increasing sequences of natural numbers. If $A_a^{\alpha_n} = \{a^{\alpha_n} m : m, n \in \mathbb{N}; a \nmid m\}$ and $A_b^{\beta_n} = \{b^{\beta_n} m : \ m, n \in \mathbb{N}; b \nmid m\}$, then*

$$d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) > d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right).$$

and

Conjecture 3.2. *Let $a, b \in \mathbb{N} - \{1\}$, $1 < \gcd(a, b) = \min\{a, b\}$.*

1. *There exist sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ fulfilling $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) > d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$.*
2. *There exist sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ fulfilling $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) < d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$.*
3. *There exist sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ fulfilling $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) = d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$.*

Conclusion

In next paper, using above mentioned results, we are ready to prove that sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ do not form a system of independent sets in case $1 < \gcd(a,b) \neq \min\{a,b\}$. In particular, we will prove that inequality $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) > d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$ holds in this case.

Finally, we will prove that in case $1 < \gcd(a,b) = \min\{a,b\}$, all three possibilities may occur and these are $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) > d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$, $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) < d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$ and also $d\left(A_a^{\alpha_n} \cap A_b^{\beta_n}\right) = d\left(A_a^{\alpha_n}\right) d\left(A_b^{\beta_n}\right)$.

Hence in case $1 < \gcd(a,b) = \min\{a,b\}$ there exist sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$, which certainly do not form a system of independent sets, however there also exist sets $A_a^{\alpha_n}$ and $A_b^{\beta_n}$ which form a system of independent sets.

References

[1] HARDY, G. H., E. M. WRIGHT. *An Introduction to the Theory of Numbers*. Oxford: Clarendon press, 1954. ISBN 978-0199219865.

[2] HOOLEY CH. On the intervals between numbers that are sums of two squares, *Journal für die reine und angewandte Mathematik*. 1974, vol. 267, pp. 207–218. ISSN 1435-5345

[3] JAHODA, P., M. PĚLUCHOVÁ. Systems of sets with multiplicative asymptotic density. *Mathematica Slovaca*. 2008, vol. 58, iss. 4, pp. 393–404. ISSN 1337-2211.

[4] JAHODA, P., M. JAHODOVÁ. On a set of asymptotic densities. *Acta Mathematica Universitatis Ostraviensis*. 2009, vol. 2008, iss. 16, pp. 21–30. ISSN 1804-1388.

[5] JAHODA, P., M. JAHODOVÁ. On systems of independent sets. *Mathematica Slovaca*. 2015, vol. 65, iss. 1, pp. 33–44. ISSN 1337-2211.

[6] JAHODOVÁ, M. *Systémy nezávislých množin*. Ostrava, 2011. Thesis. University of Ostrava. Supervisor: János Tóth.

[7] JAHODA, P. Notes on the expression of natural numbers as sum of powers. *Tatra Mountains Mathematical Publication*. 2005, vol. 34, pp. 1–11, ISSN 1210-3195.

[8] KAC, M. Statistical independence in probability, analysis and number theory. *Carus Mathematical Monographs,* . New Jersey: The Mathematical Asociation of America, 1959. LCCN 59-14986

[9] LANDAU, E. Über die Einteilung der ... Zahlen in 4 Klassen .... *Arch. Math. Phys.* 1908. vol. 13, pp. 305–312.

[10] RIEGER, G. J. Zum Satz von Landau über die Summe aus zwei Quadraten. *Journal für die reine und angewandte Mathematik*. 1970, vol. 244, pp. 198–200. ISSN 1435-5345