Teaching of Calculus – Hidden Gems Students Has Never Spot

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Abstract: The author would like to draw the reader's attention to the several elementary calculus results which are not generally well known even though they posses an intrinsic mathematical beauty. The Universal chord theorem, for example, has been repeatedly proved in the last century but never found its way into the mainstream calculus textbooks.

Keywords: Teaching, calculus, Universal chord theorem.

1 Introduction

There are several so called *mean value theorems* that are very well known and are part of a usual calculus curriculum. The most prevalent is (see also Figure 1):

Theorem 1 ([8, Theorem 5.10])**.** *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

On a picture you can see the geometric meaning of the mean value theorem. It says the function has a tangent line parallel to the straight line joining points $[a, f(a)]$ and $[b, f(b)]$.

The previous theorem is a special case of what is sometimes called a *generalized mean value theorem*:

Theorem 2 ([8, Theorem 5.9])**.** *If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$\big(f(b) - f(a)\big)g'(x) = \big(g(b) - g(a)\big)f'(x).$$

Let us mention that the case of the mean value theorem, when $f(a) = f(b)$, is known as the Rolle's theorem. On the other hand, the mean value theorem can be considered as an easy consequence of the Rolle's theorem. It's sufficient to apply it on a function:

$$f^*(x) = f(x) - \big(\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\big).$$

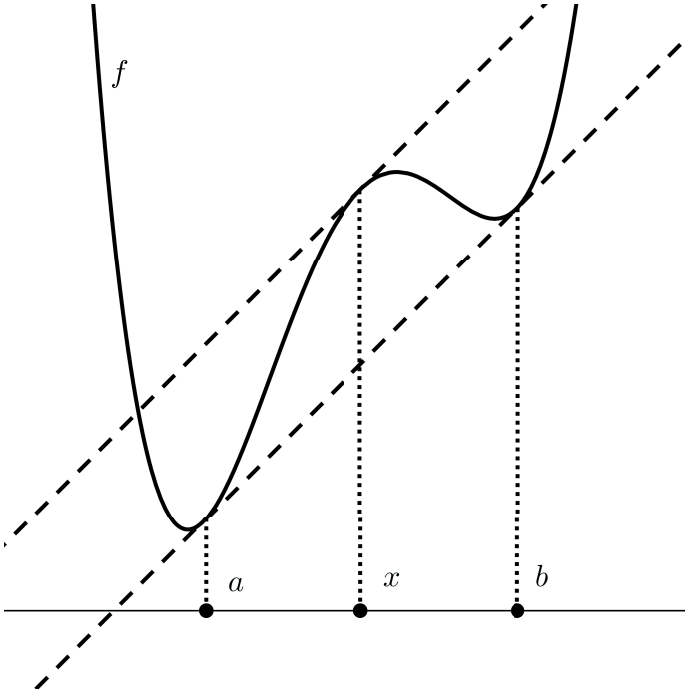

Figure 1: Mean value theorem

When differentiability is not assumed we are not allowed to talk about the tangent lines. Nevertheless, one fundamental result, called the *intermediate value theorem*, holds true:

Theorem 3 ([8, Theorem 4.23])**.** *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

2 Universal chord theorem

Instead of the tangent lines, we can employ the *chords* (see Figure 2). For any real function defined on a bounded or unbounded interval we define the *chord set* as:

$$H(f) = \{h \in [0, \infty) : f(x) = f(x + h) \text{ for some } x\}.$$

The *Universal chord theorem* of P. Lévy ([6], see also [2, p. 98]) states:

Theorem 4 (Universal chord theorem)**.** *Let f be a continuous real function defined on an interval. Then*

(i) If $h \in H(f)$, then $h/n \in H(f)$ for every positive integer n.

(ii) If a and h are positive numbers and a is not a submultiple of h, then there exists a continuous function f with $h \in H(f)$ and $a \notin H(f)$.

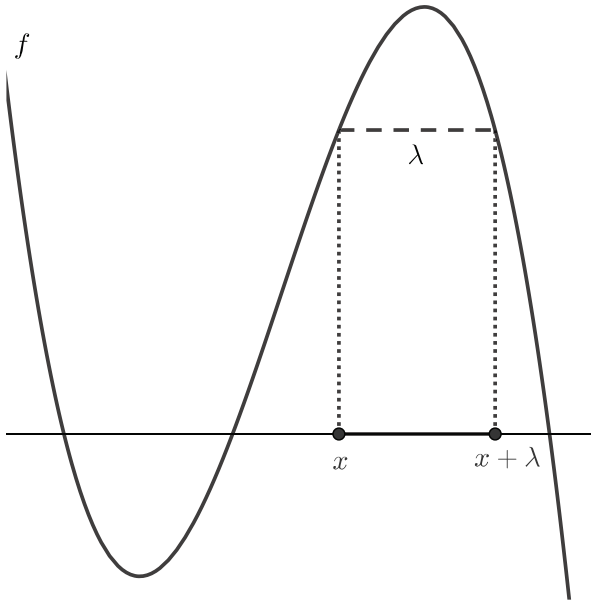

Figure 2: Chord

I furnish you with a proof and several consequences and applications. All of the results provided here are available in literature and I provide precise references too.

Proof (of the Universal chord theorem). (i) Suppose $h \in H(f)$ and $f(a) = f(a+h)$. Consider the continuous function g, defined by

$$g(x) = f\left(x + \frac{h}{n}\right) - f(x) \quad \text{for all } x \in [a, a + h - h/n].$$

If 0 is in a range of g the proof is finished. If not then g would be either positive for all x in its domain or negative for all x in the domain. That follows from the intermediate value theorem. So

$$g(a) + g\left(a + \frac{h}{n}\right) + g\left(a + \frac{2h}{n}\right) + \cdots + g\left(a + h - \frac{h}{n}\right)$$

would be either positive of negative. On the other hand:

$$\begin{aligned} &g(a) + g\left(a + \frac{h}{n}\right) + g\left(a + \frac{2h}{n}\right) + \cdots + g\left(a + h - \frac{h}{n}\right) \\ &= (f(a + h/n) - f(a)) + (f(a + 2h/n) - f(a + h/n)) + \cdots + (f(a + h) - f(a + h - h/n)) \\ &= f(a + h) - f(a) = 0 \end{aligned}$$

which is a contradiction.

(ii) We define such a function as:

$$f(x) = \sin^2\left(\frac{\pi x}{a}\right) - \frac{x}{h}\sin^2\left(\frac{\pi h}{a}\right).$$

Then $h \in H(f)$ because

$$f(0) = 0,$$
$$f(h) = \sin^2\left(\frac{\pi h}{a}\right) - \frac{h}{h}\sin^2\left(\frac{\pi h}{a}\right) = 0.$$

On the other hand, if $a \in H(f)$ there exists $x \in \mathbb{R}$ that $f(x+a) = f(x)$. Equivalently:

$$\sin^2\left(\frac{\pi(x+a)}{a}\right) - \frac{x+a}{h}\sin^2\left(\frac{\pi h}{a}\right) = \sin^2\left(\frac{\pi x}{a}\right) - \frac{x}{h}\sin^2\left(\frac{\pi h}{a}\right),$$
$$\sin^2\left(\frac{\pi x}{a} + \pi\right) - \left(\frac{x}{h} + \frac{a}{h}\right)\sin^2\left(\frac{\pi h}{a}\right) = \sin^2\left(\frac{\pi x}{a}\right) - \frac{x}{h}\sin^2\left(\frac{\pi h}{a}\right),$$
$$\frac{a}{h}\sin^2\left(\frac{\pi h}{a}\right) = 0,$$
$$\sin^2\left(\frac{\pi h}{a}\right) = 0,$$

which is a contradiction since a is not a submultiple of h.

□

We may now discuss two questions:

1. If we strengthen the assumption will we obtain reasonably stronger results?
2. If we weaken the assumptions will we obtain reasonably interesting results?

I will try to answer these questions in the rest of the paper.

3 Strengthening the assumptions

Theorem 5 ([7, Exercise 4.7.9])**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function. Then $H(f) = \mathbb{R}$.*

Proof. Let T be the period of f. Pick $\lambda > 0$. We want to prove that $\lambda \in H(f)$. Consider the continuous function

$$g(x) = f(x+\lambda) - f(x)$$

and choose $x_m, x_M \in [0,T]$ such that $f(x_m) = \min_{x\in[0,T]} f(x)$ and $f(x_M) = \max_{x\in[0,T]} f(x)$. Then $g(x_m) \geq 0$ and $g(x_M) \leq 0$. Since g is a continuous function, it sports the intermediate value property and there must exist $x_0 \in [0,T]$ such that $g(x_0) = 0$. It means:

$$0 = g(x_0) = f(x_0+\lambda) - f(x_0), \text{ hence } f(x_0+\lambda) = f(x_0) \text{ and } \lambda \in H(f).$$

□

The previous theorem can be slightly strengthen.

Theorem 6 ([7, Exercise 4.7.11]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function with period T. Then for every $\lambda \in H(f)$ there exist two different points x_1, x_2 such that $f(x_2 + \lambda) - f(x_2) = f(x_1 + \lambda) - f(x_1) = 0$.*

Proof. Let $\lambda \in H(f)$. If f is constant, the statement is clear. Assume f is non-constant. Consider the auxiliary function

$$g(x) = f(x + \lambda) - f(x).$$

The function g is T-periodic and so $g(0) = g(T)$. Also

$$\int_0^T g(x)\,dx = \int_0^T f(x + \lambda)\,dx - \int_0^T f(x)\,dx = 0.$$

and therefore g has both positive and negative values on $[0, T]$. Thus, g vanishes at least at two points in $[0, T)$, we denote these points as $x_1 \neq x_2$. It follows that

$$f(x_1 + \lambda) = f(x_1) \text{ and } f(x_2 + \lambda) = f(x_2)$$

which completes the proof. □

Theorem 7. *Let f be a continuous convex function defined on $[a, b]$ such that $f(a) = f(b)$. Then $[0, b - a] \subset H(f)$.*

Proof. If f is constant then the statement is trivially true. Assume otherwise. Let $m = \min_{x\in[a,b]} f(x)$. Then the set $\{x; f(x) = m\}$ is an preimage of a convex set by a convex function and therefore convex itself. As a preimage of a closed set it is closed as well. The closed convex sets in $\mathbb{R}$ are exactly the closed intervals. We denote $\{x; f(x) = m\} = [c_1, c_2]$.

The function f is then decreasing on $[a, c_1]$ (we denote $g_1 = f \restriction_{[a,c_1]}$) constant on $[c_1, c_2]$ and increasing on $[c_2, b]$ (we denote $g_2 = f \restriction_{[c_2,b]}$).

Clearly $[0, c_2 - c_1] \subset H(f)$. We define an auxiliary function

$$g(y) = g_2^{-1}(y) - g_1^{-1}(y), \text{ for } y \in [m, f(a)].$$

The function g is continuous and decreasing on its domain,

$$g(m) = g_2^{-1}(m) - g_1^{-1}(m) = c_2 - c_1 \text{ and } g(f(a)) = g_2^{-1}(f(b)) - g_1^{-1}(f(a)) = b - a.$$

Due to the intermediate value property $g([m, f(a)]) = [c_2 - c_1, b - a]$ and hence $[c_2 - c_1, b - a] \subset H(f)$. Employing the trivial observation $[0, c_2 - c_1] \subset H(f)$ we may conclude that $[0, b - a] \subset H(f)$.

□

Corollary 1. *The same conclusion as in Theorem 7 holds true for concave functions.*

Proof. If the function f is concave then $-f$ is convex. Then we may apply Theorem 7 on $-f$ and employ a trivial identity $H(f) = H(-f)$. □

Remark 1. The author did not find the previous theorem and its corollary anywhere in literature. These simple observations are thus probably original results of the paper in hand.

4 Weakening of assumptions

It is worth noting that the proof of the positive part of the Universal chord theorem does not actually require a continuity of f. We used the intermediate value property of g and hence we can generalize the statement a bit. Class of the functions enjoying the intermediate value property contains the continuous functions but is broader. For example, it contains also all the derivatives of the differentiable functions.

Lemma 1. *Let f be a real function defined on an interval, $h \in H(f)$ and $n \in \mathbb{N}$. If the function $x \to f(x + \frac{h}{n}) - f(x)$ enjoys the intermediate property, then $h/n \in H(f)$.*

Remark 2. It's not enough to assume the intermediate property of f, since the sum of two functions with this property does not have to enjoy it. The sum of two continuous functions is continuous as well and this fact is employed in the proof.

Another important generalization of the continuous functions are the Baire one functions.

Definition 1. Let $D \subset \mathbb{R}$. We say a function $f : D \to \mathbb{R}$ is a *Baire-one function* if it is a pointwise limit of a sequence of continuous functions, that is, if there is a sequence (f_n) of functions continuous on D such that for every $x \in D$, $f(x) = \lim_{n\to\infty} f_n(x)$.

There is an example of a function provided in [5, p. 468] given by the formula:

$$f(x) = \begin{cases} \cos\left(\frac{1}{\sin x}\right) + \frac{1}{2}(-1)^{[x/\pi]} & \text{if } x/\pi \notin \mathbb{Z} \\ \frac{1}{2}(-1)^{[x/\pi]} & \text{if } x/\pi \in \mathbb{Z}, \end{cases}$$

where $[x]$ denotes the largest integer less then or equal to x. It is a 2π-periodic Baire one function with the intermediate property which has no chord of length π. This shows that nor Theorem 4 nor Theorem 5 can be generalized dropping continuity of f in favour of the intermediate value property or being Baire one functions.

5 Possible applications

One satisfactory illustration of the Universal chord theorem's meaning is used as an initial motivation in [3]. Let me rephrase the problem. On November 16, 2013, Molly Huddle ran 37:49 for 12 kilometres and set a world record for this non-standard distance. However, Mary Keitany's world record for the half marathon (i.e., 21.1 kilometres) was 65:50 and her average pace was *higher* than in case of Molly Huddle. Does it mean that Mary Keitany must have run some 12 km subset of the race faster than 37:49? Surprisingly, because 12 is not a submultiple of 21.1 the answer is a conclusion of the Universal chord theorem and it is negative! Let us remark that in this case we use a (straightforward) generalization of the theorem for non-parallel chords. A construction of a hypothetical Mary Keitany's recordless run is provided in [3].

Conclusion

Investigation of chords of real function might be a source of many fruitful observations and statements. Many of them have elementary proofs and accessible even for the students of the first year of the university. It could be used in a special seminaries for students with deeper interest in calculus. The Universal chord theorem has also several 'popular' applications (see [3], [4]) which might engage students' attention.

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Výuka kalkulu – poklady, které nejspíš zůstanou studentům navždy skryty

Abstrakt: Rádi bychom čtenáře seznámili s některými výsledky kalkulu, které byť jsou dokazatelné elementárními prostředky a je jim vlastní matematická elegance, tak nejsou všeobecně známé. Zejména se zmíníme o všeobecné tětivové větě, která byla dokázána na začátku 20. století, ale nenašla si cestu do klasických učebnic kalkulu, tak byla několikrát nezávisle znovudokázána.

Klíčová slova: Výuka, kalkulus, všeobecná tětivová věta.